Holography for Black Holes in General Relativity and Beyond

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Abstract

Both extremal and non-extremal Kerr black holes have been considered to be holographically dual to two-dimensional (2D) conformal field theories (CFTs). In this thesis, we study the holography to the case of a rotating Janis-Newman-Winicour (JNW) black holes, a rotating Brans-Dicke-Kerr (BDK) black hole, and an asymptotically anti-de Sitter (AdS) rotating charged black holes in $f(T)$ gravity, where $f(T) = T + \alpha T^2$, where α is a constant. Firstly, we find that the rotating JNW solution does not satisfy the Einstein field equation. Thus, we could not establish a well-defined Kerr/CFT correspondence in this theory. Secondly, we find that the scalar wave equation in the background of BDK black hole is not separable. The existence of the $SL(2,R)_L \times SL(2,R)_R$ symmetry can be found in the radial equation of the scalar probe around the non-extremal black hole. Therefore, the inseparability of the scalar wave equation eliminates the possibility of any holography aspect for BDK black hole. Thirdly, we find that the scalar wave radial equation at the near-horizon region implies the existence of the 2D conformal symmetries. We note that the 2π identification of the azimuthal angle ϕ in the black hole line element, corresponds to a spontaneous breaking of the conformal symmetry by left and right temperatures T_L and T_R, respectively. We show that choosing proper central charges for the dual CFT, we produce exactly the macroscopic Bekenstein-Hawking entropy from the microscopic Cardy entropy for the dual CFT. These observations suggest that the rotating charged AdS black hole in $f(T)$ gravity is dual to a 2D CFT at finite temperatures T_L and T_R for a specific value of mass M, rotational, charge, and $f(T)$ parameters, Ω, Q, and $|\alpha|$, respectively.

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Canisius Bernard
Saskatoon, Canada

To my parents

"Ipsa scientia potestas est."

— *Sir Francis Bacon, Meditationes sacrae*

Contents

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List of Abbreviations

AdS	Anti-de Sitter
BD	Brans-Dicke
BDK	Brans-Dicke-Kerr
CFT	Conformal Field Theory
GR	General Relativity
JNW	Janis-Newman-Winicour
KG	Klein-Gordon
NHEK	Near-horizon Extremal Kerr
QFT	Quantum Field Theory
SCT	Special Conformal Transformation
SL	Special Linear
SO	Special Orthogonal
TEGR	Teleparallel Equivalent General Relativity

1 Introduction

Black holes are fundamental objects and a spectacular prediction of Einstein's theory of gravity that is believed to exist in the Universe. These exotic objects can be studied both theoretically and experimentally. A black hole is a spacetime singularity that has a boundary called the event horizon. A black hole is a region where gravity is vigorous enough to pull light back such that it cannot escape from inside the horizon. For quite a long time, people questioned the existence of black holes. We could only see indirect evidence of them by observing the behavior of stars that orbit unseeable objects. Einstein himself claimed that stars could not collapse under gravity since matter could not be compressed beyond a certain point [2]. After the centennial of general relativity, there is stunning evidence that black holes exist out there in the Heavens. In 2019, the Event Horizon Telescope collaboration unveiled the first-ever picture of a supermassive black hole with the same mass as 6.5 billion Suns in the heart of Messier 87 galaxy more than 50 million light-years away from Earth. They published their findings in six separate articles [1, 3, 4, 5, 6, 7]. According to general relativity, a geometric theory of gravitation published by Albert Einstein in 1915, a black hole can be characterized by only three physical quantities, which are its mass, angular momentum, and electric charges. These characterizations are known as no-hair theorems.

Figure 1.1: Event Horizon Telescope image of the Messier 87 black hole shadow. This figure is taken from Ref. [1].

Einstein's general relativity cannot be defined at a singularity since it is a classical theory. For very short distances, there are randomness and uncertainty consequences that are unavoidable. In this manner, we need a quantum theory of gravity. Gerard 't Hooft, followed by Leonard Susskind [8, 9], suggested that the merger of general relativity and quantum mechanics requires information in a three-dimensional (3D) world that can be projected on a 2D space, much like a hologram. This is known as the principle of holography. Asymptotic symmetries make a natural appearance in holography, the proposed equivalence between a quantum gravity theory on a given spacetime manifold $\mathbb{M}$ and a field theory living on its boundary $\partial\mathbb{M}$. More precisely, they should correspond to global symmetries of the dual field theory. The type of the field theory depends only on the asymptotic structure of the manifold, not on the gravitational theory. The Noether theorem states that continuous global symmetries lead to local conserved (Noether) currents. In gauge theory, the currents associated to gauge transformations vanish since gauge symmetries are not physical symmetries. However, if the manifold on which the gauge theory is defined has a boundary and the gauge parameter does not vanish on the boundary, then the associated conserved charge can be non-zero. Such gauge transformations that do not vanish at infinity (boundary) are known as asymptotic symmetries [10].

Holographic principle idea was inspired by the concept of the entropy of the black holes. Black holes obey the fundamental laws of physics, for example, the law of conservation of energy and the second law of thermodynamics. Classically, black holes are often thought of as a dead thermodynamics object. They have no entropy. This paradox was resolved by Jacob Bekenstein, further clarified by Stephen Hawking and many others [11, 12, 13, 14, 15, 16, 17]. The essential result which is known as the Bekenstein-Hawking entropy formula, states that the entropy of a black hole S is proportional to the area A of its event horizon:

$$S = \frac{\pi A k_B c^3}{2hG}, \tag{1.1}$$

where k_B is the Boltzmann constant, c is the speed of light in the vacuum, h is the Planck constant, and G is the Newton gravitational constant. The example of the holographic principle was proposed by Maldacena, in his famous anti-de Sitter/conformal field theory (AdS/CFT) correspondence conjecture [18]. The AdS/CFT or sometimes also called the gauge/gravity duality, relates a quantum field theory (QFT), a theoretical framework that combines classical field theory, special relativity and quantum mechanics, in $N-1$ dimensions and a quantum gravity theory in N dimensions. The degrees of freedom of the CFT, live on the boundary of the AdS spacetime.

We do not live in a Universe with a negative cosmological constant (AdS spacetime). The current best fit model is a flat Universe with a slight positive curvature due to the cosmological constant (dark energy) causing its expansion. For this reason, an instinctive question to be asked is, how can we get the benefit of holographic ideas to the *real* world? The idea of AdS/CFT correspondence was applied to the case of extremal rotating black holes, namely, the Kerr/CFT correspondence, which was proposed by Guica et al. [19]. The correspondence states that the physics of an extremal Kerr black hole, a black hole that has a rotational parameter of unity, can be described by a 2D CFT, living on the near-horizon region of the black holes. The correspondence was established by showing that one can microscopically reproduce the Bekenstein-Hawking entropy, using the CFT Cardy entropy formula. For extremal black holes, the near-horizon geometry is an AdS space with isometries that can be extended to Virasoro algebra. Hence, the conformal symmetries appear at the near-horizon region of extremal black holes. As one would expect, the Kerr/CFT correspondence is not only a peculiar property of extremal black holes but also non-extremal Kerr black holes.

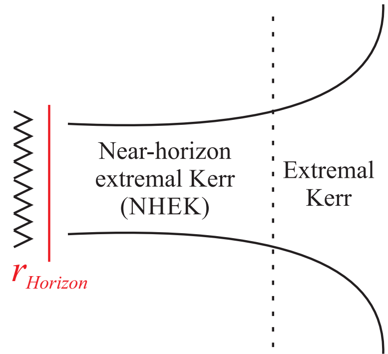


Figure 1.2: Extremal Kerr black hole illustration.

However, at the near-horizon region of the non-extremal Kerr black holes, we cannot indicate any conformal symmetries. In other words, the conformal symmetries are not the symmetries of non-extremal Kerr black hole geometry (as they are for the case of the extremal Kerr black holes). The problem is that far away from the extremal limit, the near-horizon extremal Kerr (NHEK) geometry disappears, and the near-horizon geometry is a Rindler space. Note that CFT is not associated with a Rindler space. The near-horizon geometry such as NHEK or AdS with a conformal symmetry group is not a necessary condition for interactions to show the conformal invariance. The *hidden* conformal symmetries can be revealed by looking at the solution space of the radial part of the Klein-Gordon wave equation for a massless scalar probe in the near-horizon region of the Kerr black hole [20]. In this case, the radial equation can be written as the $SL(2,R)_L \times SL(2,R)_R$ Casimir eigen-equation. Subsequently, the Kerr/CFT correspondence can be established by matching the microscopic CFT Cardy entropy to the macroscopic Bekenstein-Hawking entropy of the Kerr black holes with general angular momentum and mass parameters. The correspondence has been studied for several black hole solutions, for instance, in Refs. [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35].

In this thesis, we extend the hidden conformal symmetry discussions to the rotating black hole solution, namely, the Janis-Newman-Winicour (JNW) and Brans-Dicke-Kerr (BDK) black holes. The rotating JNW spacetime describes a rotating mass in Einstein's theory of gravity minimally coupled to a real scalar field. This spacetime is the simplest generalization of the Kerr spacetime, and bizarrely it does not possess any event horizon. It is a surface-like singularity where curvature invariants diverge. The BDK spacetime is a stationary axisymmetric solution in the vacuum Brans-Dicke (BD) theory from the Kerr solution in vacuum Einstein's theory. This Kerr-like solution has a scalar field which is singular at the horizon. The BD theory is a scalar-tensor extension to Einstein's general relativity.

The traditional theory of gravity, based on the Riemannian geometry, has been extended through several gravity theories. One of them is the teleparallel equivalent general relativity (TEGR) theory, where its Lagrangian density, the Ricci scalar R, is replaced by the teleparallel torsion scalar T. Moreover, the TEGR has been generalized to $f(T)$ gravity by replacing the torsion scalar T, with an arbitrary function of T, such as $f(T)$. Finding an exact black hole solution in this extension theory is not a straightforward task. The charged black hole solutions in $f(T)$ gravity in three and higher spacetime dimensions was found first in Ref. [36, 37]. In [38], Awad et al. find an asymptotically rotating charged AdS black hole solution in all dimensions within the $f(T)$-Maxwell theory with a negative cosmological constant. This black hole possesses a cylindrical event horizon.

Inspired by the existence of dual CFT for almost all generic rotating black holes, we investigate the holographic description of the black hole in Ref. [38]. We adopt the technique in which we find the hidden conformal symmetry by the scalar scattering off a black hole. The novel feature in this study is to consider a black hole background in the gravity theory beyond Einstein's gravity.

The outline of this thesis is organized as follows. Chapter 2 is divided into two parts. First, we review the Einstein gravity theory and Kerr black holes. Starting from the Einstein-Hilbert action, we obtain the vacuum Einstein field equations using the variational principle approach. We briefly review the Kerr black hole and its important properties. We also present the thermodynamics aspect of Kerr black holes. Second, we review the teleparallel gravity by introducing torsion tensors, TEGR, and its $f(T)$ extension. Then we discuss the rotating black hole solution in the quadratic $f(T)$ gravity theory.

In chapter 3, we review the CFT. Starting from CFT in $N > 2$ dimensions and then we discuss the conformal group in $N = 2$ dimensions and the energy-momentum tensor. For 2D CFT, we also consider the corresponding conformal group and define the central charge in the Witt algebra that leads to the Virasoro algebra. Next, we study the primary field, energy-momentum tensor, and derive the partition function for CFT on the torus to obtain the Cardy entropy formula.

In chapter 4, we briefly review the Kerr/CFT correspondence conjecture to find the central charge of extremal Kerr black holes from the symmetry of spacetime in the NHEK geometry. One can reproduce the macroscopic Bekenstein-Hawking entropy formula of Kerr black holes using the Cardy entropy formula in 2D CFT. We also discuss the hidden conformal symmetry of Kerr black holes, which establishes the Kerr/CFT correspondence for non-extremal Kerr black holes.

In chapter 5, inspired by the Kerr/CFT correspondence idea, we apply this technique to investigate the holographic aspect of black holes in the theory beyond Einstein gravity, i.e., scalar-tensor gravity theory and $f(T)$ gravity theory. Firstly, we present evidence that a rotating JNW black hole solution does not satisfy Einstein field equations. Secondly, we find that one cannot establish the Kerr/CFT correspondence in the background of BDK spacetime. Thirdly, we find that the 4D rotating charged AdS black holes in quadratic $f(T)$ gravity can be considered holographically dual to the 2D CFT. In the final chapter, we conclude our works and address some open questions for future works. In this thesis, we use the Planck units, in which $c = G = \hbar = k_B = 1$.

2 Black Holes in Einstein Gravity and Beyond

2.1 Einstein gravity

Einstein suggests that gravity is a manifestation of spacetime curvature induced by the presence of matter and energy. Therefore, we must obtain a set of equations that describe how the curved spacetime at any event is related to the matter distribution at that event. These will be the famous gravitational field equations, or Einstein's field equations[1]:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.1}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, and R is the Ricci scalar or curvature scalar. Field equations (2.1) relate spacetime curvature to its source, the energy-momentum of matter tensor $T_{\mu\nu}$. Einstein equations are a set of non-linear second-order partial differential equations. Thus, to find a general solution for an arbitrary matter distribution is a very difficult task. It is easier, however, to solve the problem if we find special solutions, for instance those representing spacetimes possessing symmetries. In 1916, the first spherically symmetric exact solution of Einstein's field equations was found by a German physicist, Karl Schwarzschild. Despite serving in the World War I as a lieutenant in the artillery and suffering from a rare and painful skin disease called pemphigus, Schwarzschild published three outstanding articles, two on relativity theory, and one on quantum theory. His article on general relativity produced the first spherically symmetric exact solution to Einstein's field equation in vacuum

$$R_{\mu\nu} = 0, \tag{2.2}$$

and the famous solution that now bears his name, the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{2.3}$$

The Schwarzschild geometry describes the spacetime geometry outside a spherically symmetric massive body characterized only by its mass M. Most real astrophysical objects, however, are rotating. It took almost fifty years after the discovery of Schwarzschild solution for Roy Kerr to obtain an acceptable solution of Einstein's field equations outside of a rotating massive body in 1963. Unlike Schwarzschild, Kerr spacetime is characterized not only by its mass but also its angular momentum. In this chapter we first derive the Einstein's field equations using the variational principle and then review the vacuum solution of Einstein

[1] We have restored the factor of Newton's universal gravitational constant G and the speed of light in the vacuum c for more clarity.

equations, namely, the Kerr solution and its important properties. The main references for this chapter are [39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49]

2.1.1 A variational approach to general relativity

In this section, we present an introduction of variational principles and the Lagrangian formalism. The physical meaning of a theory can clearly be understood by expressing the theory in terms of variational principle. The purpose is to derive the field equation of general relativity. This requires us to consider some general aspects of classical field theory in both flat and curved spacetime.

The free Klein-Gordon field theory

Before reviewing the action that describes gravity in GR, we would like to review the action of a free Klein-Gordon field $\phi(x)$. We assume that there exists an action

$$S = \int_{t_i}^{t_f} dtL = \int_{t_i}^{t_f} dtd^3x\mathcal{L} = \int_{t_i}^{t_f} d^4x\mathcal{L}, \tag{2.4}$$

where we write the Lagrangian L in terms of the Lagrangian density $\mathcal{L}$. In addition, we also assume that the Lagrangian density depends on the field and its first derivatives:

$$\mathcal{L} = \mathcal{L}\left(\phi(x), \partial_\mu\phi(x)\right). \tag{2.5}$$

Generally, a Lagrangian density can depend on higher order derivatives but for *equations* with at most second order derivatives, the Lagrangian density can depend at the most on the first order derivatives of the field. Next, we examine conditions which make the action stationary if we change the fields as

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x), \tag{2.6}$$

with the boundary conditions

$$\delta\phi(\mathbf{x}, t_i) = \delta\phi(\mathbf{x}, t_f) = 0. \tag{2.7}$$

Under an infinitesimal change, the action (2.4) becomes

$$\begin{aligned}
\delta S &= \int_{t_i}^{t_f} d^4x\delta\mathcal{L} \\
&= \int_{t_i}^{t_f} d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi(x)}\delta\phi(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\delta(\partial_\mu\phi(x))\right] \\
&= \int_{t_i}^{t_f} d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi(x)}\delta\phi(x) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\partial_\mu\delta\phi(x)\right] \\
&= \int_{t_i}^{t_f} d^4x\left\{\frac{\partial\mathcal{L}}{\partial\phi(x)}\delta\phi(x) + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\delta\phi(x)\right] - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\right\} \\
&= \int_{t_i}^{t_f} d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi(x)} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))}\right]\delta\phi(x) + \int d^3x\frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)}\delta\phi(x)\Big|_{t_i}^{t_f}.
\end{aligned} \tag{2.8}$$

In above derivation, we apply Gauss' theorem and the vanishing field at spatial infinity, $\lim_{|x|\to\infty}\phi(x) \to 0$, to simplify the surface term (the last term of Eq. (2.8)). We also apply the identity in which $\delta(\partial_\mu\phi(x)) =$

$\partial_\mu(\delta\phi(x))$ in the above calculation. Moreover, we note that the surface term of Eq. (2.8) vanishes due to the boundary condition (2.7). Therefore, the change in the action under an arbitrary, infinitesimal change can be written as

$$\delta S = \int_{t_i}^{t_f} d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \right] \delta\phi(x). \tag{2.9}$$

We can see that the change of the action under an arbitrary, infinitesimal, change in the field will be stationary i.e. $\delta S = 0$ will be stationary only if

$$\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} = 0. \tag{2.10}$$

This equation is known as the Euler-Lagrange equation which is related with the action or the Lagrangian density.

Now we consider the Lagrangian for a Klein-Gordon field given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \frac{1}{2}m^2\phi^2(x), \tag{2.11}$$

which gives

$$\frac{\partial \mathcal{L}}{\partial \phi(x)} = -m^2\phi(x), \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} = \partial^\mu \phi(x). \tag{2.12}$$

In this case, the Euler-Lagrange equation gives

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi(x) = 0. \tag{2.13}$$

This means, the Lagrangian density (2.11) gives the Klein-Gordon equation for free, real scalar fields as its field equation. Moreover, we can consider the Lagrangian density of a free massless Klein-Gordon equation in curved space

$$\mathcal{L} = \sqrt{-g}\partial_\mu \phi(x)\partial^\mu \phi(x), \tag{2.14}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$. The field equation for the above Lagrangian density gives

$$\partial_\mu(\sqrt{-g}\partial^\mu \phi(x)) = \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu \phi(x)) = 0, \tag{2.15}$$

which is the equation of motion for massless Klein-Gordon fields in curved spacetime. This scalar field equation will be studied later in chapters four and five.

The Einstein-Hilbert action

After deriving the scalar field theory from the variational principle, we use our experience to construct an action for gravitation resulting in the Einstein's field equation of general relativity. First, we limit ourselves to the vacuum case in which neither matter nor energy are outside of a massive body that curves the spacetime. To construct an action for gravity, we define a Lagrangian which is a scalar under general coordinate transformations and depends on the metric tensor $g_{\mu\nu}$. The action is known as the Einstein-Hilbert action:

$$S = \int d^4x R\sqrt{-g}, \tag{2.16}$$

where $R = g_{\mu\nu} R^{\mu\nu}$ is the Ricci scalar and the Ricci tensor is defined as

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\alpha\beta} - \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta}, \tag{2.17}$$

and $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol of the second kind:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu} \right). \tag{2.18}$$

Thus, the Euler-Lagrange equations take the form

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \partial_\sigma \left[\frac{\partial \mathcal{L}}{\partial(\partial_\sigma g_{\mu\nu})} \right] + \partial_\rho \partial_\sigma \left[\frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\sigma g_{\mu\nu})} \right] = 0. \tag{2.19}$$

Since the work of calculating each term in the Euler-Lagrange equation involve an extensive amount of algebra, we shall find another approach to derive the Einstein equations. We will consider the variation in the action resulting from a variation in the metric tensor:

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \tag{2.20}$$

in which, $\delta g_{\mu\nu}$ and its first derivatives go away on the boundary $\partial \mathcal{R}$ of the region $\mathcal{R}$. Note that we have the relation $g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu$ where δ^μ_ν is invariant under a variation. Thus, to first order in the variation:

$$\delta g^{\mu\rho} g_{\rho\nu} + g^{\mu\rho} \delta g_{\rho\nu} = 0. \tag{2.21}$$

We multiply above equation by $g^{\nu\sigma}$ and with some rearrangement we get

$$\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma}, \tag{2.22}$$

which is a useful relation for the later calculation.

Next, we write the first order variation of the Einstein-Hilbert action as

$$\begin{aligned} \delta S &= \int_{\mathcal{R}} d^4x \sqrt{-g}\, \delta g^{\mu\nu} R_{\mu\nu} + \int_{\mathcal{R}} d^4x \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} + \int_{\mathcal{R}} d^4x\, \delta(\sqrt{-g}) g^{\mu\nu} R_{\mu\nu} \\ &\equiv \delta S_1 + \delta S_2 + \delta S_3. \end{aligned} \tag{2.23}$$

In order to derive the field equation, we have to factor $\delta g_{\mu\nu}$ out in δS_2 and δS_3. Firstly, we focus on δS_2 and write $\delta R_{\mu\nu}$ in terms of $\delta g^{\mu\nu}$. Since the Ricci tensor can be obtained by contracting the Riemann tensor, we will determine the variation of the Riemann tensor instead. The Riemann curvature tensor is given by

$$R^\sigma_{\mu\nu\rho} = \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\rho \Gamma^\sigma_{\mu\nu} + \Gamma^\lambda_{\mu\rho} \Gamma^\sigma_{\lambda\nu} - \Gamma^\lambda_{\mu\nu} \Gamma^\sigma_{\lambda\rho}. \tag{2.24}$$

Let us consider the variation in the affine connection coefficients

$$\Gamma^\sigma_{\mu\nu} \to \Gamma^\sigma_{\mu\nu} + \delta \Gamma^\sigma_{\mu\nu}. \tag{2.25}$$

Before deriving the variation of the Riemann tensor, we would like to discuss the local geodesic coordinate systems. In such coordinates, every component of the Christoffel symbols vanishes. Let us start by considering that the Christoffel symbols have non-vanishing components at an arbitrary point $\tilde{P}$, initially, in a coordinate system. Then, we write the following transformation:

$$P'^{\alpha} = P^{\alpha} - \tilde{P}^{\alpha} + \frac{1}{2}\Gamma^{\alpha}_{\mu\nu}(\tilde{P})(P^{\mu} - \tilde{P}^{\mu})(P^{\nu} - \tilde{P}^{\nu}). \tag{2.26}$$

Using the relation $\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}$, the transformation (2.26) results in

$$\frac{\partial^2 P'^{\alpha}}{\partial P^{\mu}\partial P^{\nu}}\Big|_{P=\tilde{P}} = -\Gamma^{\alpha}_{\mu\nu}(\tilde{P}). \tag{2.27}$$

In two different coordinates, let us say from P^{μ} to P'^{μ}, the Christoffel symbols transform as

$$\Gamma'^{\alpha}_{\mu\nu} = \frac{\partial P'^{\alpha}}{\partial P^{\beta}}\frac{\partial P^{\rho}}{\partial P'^{\mu}}\frac{\partial P^{\sigma}}{\partial' P^{\nu}}\Gamma^{\beta}_{\rho\sigma} + \frac{\partial P'^{\alpha}}{\partial P^{\beta}}\frac{\partial^2 P'^{\beta}}{\partial P'^{\mu}\partial P'^{\nu}}. \tag{2.28}$$

It is worth noting that the appearance of the second term in Eq. (2.28) shows that the Christoffel symbol does not transform properly as a tensor. Thus, the Christoffel symbol is not a tensor. In the geodesic coordinate $P'\alpha$, the transformed Christoffel symbol vanishes at $\tilde{P}$:

$$\Gamma'^{\alpha}_{\mu\nu}(\tilde{P}) = \delta^{\alpha}_{\beta}\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}\Gamma^{\beta}_{\rho\sigma}(\tilde{P}) - \delta^{\alpha}_{\beta}\Gamma^{\beta}_{\mu\nu}(\tilde{P}) = 0. \tag{2.29}$$

At the point $\tilde{P}$ we have

$$\delta R^{\sigma}_{\mu\nu\rho} = \partial_{\nu}(\delta\Gamma^{\sigma}_{\mu\rho}) - \partial_{\rho}(\delta\Gamma^{\sigma}_{\mu\nu}). \tag{2.30}$$

Furthermore, partial derivatives and covariant derivatives coincide at point $\tilde{P}$. In that case

$$\delta R^{\sigma}_{\mu\nu\rho} = \nabla_{\nu}(\delta\Gamma^{\sigma}_{\mu\rho}) - \nabla_{\rho}(\delta\Gamma^{\sigma}_{\mu\nu}). \tag{2.31}$$

Since the quantities on the right-hand side are tensors, Eq. (2.31) is valid not only in a local geodesic coordinate system at point $\tilde{P}$, but also in any arbitrary coordinate system. Equation (2.31) is known as the Palatini equation. By contracting σ and ρ in Eq. (2.31), we get the variation of the Ricci tensor:

$$\delta R_{\mu\nu} = \nabla_{\nu}(\delta\Gamma^{\sigma}_{\mu\sigma}) - \nabla_{\sigma}(\delta\Gamma^{\sigma}_{\mu\nu}). \tag{2.32}$$

Thus, we can rewrite δS_2 as

$$\begin{aligned} \delta S_2 &= \int_{\mathcal{R}} d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \\ &= \int_{\mathcal{R}} d^4x\sqrt{-g}g^{\mu\nu}\left[\nabla_{\nu}(\delta\Gamma^{\sigma}_{\mu\sigma}) - \nabla_{\sigma}(\delta\Gamma^{\sigma}_{\mu\nu})\right] \\ &= \int_{\mathcal{R}} d^4x\sqrt{-g}\nabla_{\nu}\left(g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\sigma} - g^{\mu\sigma}\delta\Gamma^{\nu}_{\mu\sigma}\right), \end{aligned} \tag{2.33}$$

Due to the fact that the covariant derivative of the metric tensor vanishes, we get $\delta S_2 = 0$.

Secondly, let us focus on δS_3 in which we have to write $\delta\sqrt{-g}$ in terms of $\delta g^{\mu\nu}$. Recalling that g is the determinant of tensor metric $g_{\mu\nu}$, we note that

$$\frac{\delta g}{\delta g_{\mu\nu}} = g g^{\mu\nu}, \tag{2.34}$$

or equivalently

$$\delta g = -g g_{\mu\nu} \delta g^{\mu\nu}. \tag{2.35}$$

Moreover, using the relation $g_{\mu\sigma} g^{\sigma\mu} = \delta^{\nu}_{\mu}$, we can write the variation of the metric tensor as

$$\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta}. \tag{2.36}$$

Thus, we have

$$\delta\sqrt{-g} = -\frac{1}{2}(-g)^{1/2}\delta g = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}. \tag{2.37}$$

Substituting above result to δS_3 and remembering $\delta S_2 = 0$, the Einstein-Hilbert action can be written as

$$\begin{aligned}
\delta S &= \delta S_1 + \delta S_2 + \delta S_3 \\
&= \int_{\mathcal{R}} d^4x \sqrt{-g}\delta g^{\mu\nu} R_{\mu\nu} + 0 + \int_{\mathcal{R}} d^4x \left(-\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}\right) g^{\mu\nu} R_{\mu\nu} \\
&= \int_{\mathcal{R}} d^4x \sqrt{-g}\delta g^{\mu\nu} R_{\mu\nu} + \int_{\mathcal{R}} d^4x \sqrt{-g}\delta g^{\mu\nu} \left(-\frac{1}{2} g_{\mu\nu} R\right) \\
&= \int_{\mathcal{R}} d^4x \sqrt{-g}\delta g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right).
\end{aligned} \tag{2.38}$$

By demanding that the action is stationary, $\delta S = 0$, and $\delta g^{\mu\nu}$ is arbitrary, we get the Einstein field equation in vacuum:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \tag{2.39}$$

2.1.2 Kerr black holes

In 1916, Karl Schwarzschild discovered an exact solution of the vacuum Einstein equation (2.39) which describes an empty spacetime outside of a static massive object. Since then, many attempts to find a rotating solution of the vacuum Einstein equation have been made. In 1963, it was Roy Kerr who first derived an asymptotically flat solution of the vacuum Einstein equations outside of a rotating massive body to generalize the static spacetime solution. Asymptotically flat means that the spacetime must be flat at infinity. In other words, at a very far away distance to the gravitational source, an observer should not feel gravity. In this subsection, we will discuss some important properties of a Kerr black hole.

Briefly stated, a black hole is a region where gravity is very strong such that nothing can escape from it, not even light. This region of spacetime is separated from infinity by an event horizon. A black hole is a result of the gravitational collapse of a massive star. Surprisingly, black holes are simply characterized by a small number of parameters. The no-hair theorems state that the geometry of black holes is determined by three parameters: mass M, angular momentum J, and electric charge Q. In an astrophysical situation, significant Q is not expected. The Kerr black hole is an axially symmetric black hole. It is rotating about one axis only, which is the angular momentum axis. Thus, a Kerr black hole is characterized by only two parameters, M, and J. The angular momentum has dimensions of m^2. Therefore we define the rotational parameter

$$a = \frac{J}{M}, \tag{2.40}$$

which has the same dimension as M. The line element i.e the spacetime interval of a Kerr black hole is

$$ds^2 = \frac{\Sigma}{\Delta}dr^2 - \frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - adt\right]^2, \tag{2.41}$$

where

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2\cos^2\theta. \tag{2.42}$$

The coordinates of metric (2.41) are called the Boyer-Lindquist coordinates (t, r, θ, ϕ). The non-vanishing components of the Kerr contravariant tensor metric are

$$g^{tt} = \frac{\Delta a^2\sin^2\theta - \left(r^2+a^2\right)^2}{\Sigma}, \quad g^{rr} = \frac{\Delta}{\Sigma}, \quad g^{\theta\theta} = \frac{1}{\Sigma}, \quad g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\Delta\Sigma\sin^2\theta}, \quad g^{t\phi} = g^{\phi t} = -\frac{2Mar}{\Delta\Sigma} \tag{2.43}$$

Several important properties of Kerr geometry are:

- *Asymptotically flat.* In the limit of $r \to \infty$, the Kerr geometry approaches the geometry of the Minkowski flat spacetime far away from the black hole:

$$ds^2 = -dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{2.44}$$

- *Stationary and axisymmetric.* The line element (2.41) is independent of t (stationary) and ϕ (axisymmetric). In other words, it has two Killing vectors that correspond to these symmetries which are:

$$\xi^\mu = (1,0,0,0), \text{ (stationary)}, \quad \eta^\mu = (0,0,0,1), \text{ (axisymmetric)}, \tag{2.45}$$

  where usually the components are given in the order (t, r, θ, ϕ).

- *Schwarzschild when not rotating.* When the rotational parameter a vanishes, the Kerr geometry reduces to the Schwarzschild geometry:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{2.46}$$

- *Coordinate singularities, real singularities and event horizon.* The Kerr line element is singular if $\Sigma = 0$ and if $\Delta = 0$. The real singularity, a place of infinite spacetime curvature, happens when $r = 0$ and $\theta = \pi/2$. Assuming $a \leq M$, the quantity Δ is zero at the radii

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \tag{2.47}$$

  Singularities at these two radii are called coordinate singularities. We can transform the line element (2.41) to new coordinates, such as Eddington-Finkelstein, Kruskal-Szekeres, and Gullstrand-Painlevé, where the line element is not singular at these radii. The outer horizon r_+ is the horizon that makes the Kerr metric a black hole. For the Schwarzschild black hole (2.46), there is only one event horizon:

$$r = 2M. \tag{2.48}$$

Not every value of M and a correspond to a Kerr black hole. The outer horizon r_+ only exists if $a \leq M$. The angular momentum of a black hole is restricted by its mass squared. Black holes that have the limit $a = M$ or $J = M^2$ are called extremal Kerr black holes.

Event horizon and singularity

A black hole is a region in spacetime where gravity is so strong that no particle or even electromagnetic wave can escape from it. The boundary where nothing can escape is called the event horizon. In other words, it is a hypersurface separating the spacetime points that are connected to infinity by a timelike path from those spacetime points that are not. Here, infinity means asymptotically flat. The event horizon is at $g^{rr} = 0$ for which $g^{rr} = \frac{\Delta}{\Sigma} = 0$. Solving $\Delta = 0$ for r gives

$$r_{\pm} = r_{\text{horizon}} = M \pm \sqrt{M^2 - a^2}. \tag{2.49}$$

The surfaces $r_{\pm}$ are called the outer and inner horizons of the Kerr black hole. The outer horizon r_+ is the physical boundary where nothing can escape and the inner horizon r_- is not physically observable. The Kerr metric is stationary. It does not depend explicitly on time t. The outer horizon (2.49) is also independent of t and r. Any surface that is independent of t and r has a metric in which the line element comes from (2.41) with $dt = dr = 0$:

$$dl^2 = \frac{(r^2 + a^2)^2 - a^2\Delta}{\Sigma} \sin^2\theta d\phi^2 + \Sigma d\theta^2. \tag{2.50}$$

The area of the surface can be calculated by

$$A(r) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{(r^2 + a^2)^2 - a^2\Delta} \sin\theta, \tag{2.51}$$

where the area of a unit sphere is

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta = 4\pi. \tag{2.52}$$

Thus we get

$$A(r) = 4\pi\sqrt{(r^2 + a^2)^2 - a^2\Delta}. \tag{2.53}$$

The horizon r_+ is defined by $\Delta = 0$. Therefore we get

$$A(r_+) = 4\pi({r_+}^2 + a^2) = 8\pi M r_+. \tag{2.54}$$

As mentioned before, Kerr black holes have two event horizons r_+ and r_-. These two horizons show that the coordinate system that we used is incapable of allowing a non-divergent metric (2.41) at all points except at $r = 0$. To make sure that the singularity at the horizons r_+ and r_- is just a coordinate singularity, we calculate a quantity named the Kretschmann scalar:

$$K = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}. \tag{2.55}$$

In principal, the definition of the Kretschmann scalar is simple but in practice, the algebraic computation is hard. The result of the Kretschmann scalar for Kerr spacetime, considering the breathtaking magnitude of the calculation that is required for its evaluation, reads

$$K = \frac{48M^2(r^2 - a^2\cos^2\theta)\left[(r^2 + a^2\cos^2\theta)^2 - 16r^2a^2\cos^2\theta\right]}{(r^2 + a^2\cos^2\theta)^6}. \tag{2.56}$$

Finite values of the Kretschmann scalar at r_+ and r_- show that the singularity at both horizons is just a coordinate singularity. Meanwhile, at $r = 0$ and $\theta = \pi/2$, the Kretchmann scalar diverges, showing that we encounter a true singularity.

At $r = 0$, Kerr black holes have a ring-shaped singularity instead of a point singularity as in Schwarzschild black holes. This can be shown from the coordinate transformation between the Kerr line element in Boyer-Lindquist coordinates and the Kerr line element in Cartesian coordinates. Applying $M \to 0$ limit to the Kerr metric gives

$$ds^2 = -dt^2 + \frac{\Sigma}{r^2 + a^2} dr^2 + \Sigma d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2. \tag{2.57}$$

The above metric is simply the flat Minkowski line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{2.58}$$

in oblate spheroidal coordinates. The coordinate transformations between them are

$$x = \sqrt{r^2 + a^2} \sin \theta \cos \phi, \tag{2.59}$$

$$y = \sqrt{r^2 + a^2} \sin \theta \sin \phi, \tag{2.60}$$

$$z = r \cos \theta. \tag{2.61}$$

Substituting $r = 0$ and $\theta = \pi/2$ to (2.59)–(2.61) resulting

$$x^2 + y^2 = a^2, \tag{2.62}$$

which is simply a circle equation with radius a on the Cartesian coordinate. The fact that the Kretschmann scalar of the Kerr spacetime diverges along the circle (2.62) suggests that the true singularity of Kerr black holes has a ring-shaped form with radius a as it is illustrated in Fig. 2.1.

Killing horizons and the ergoregion

If a Killing vector field χ^μ is null along some null hypersurface Σ, we say that Σ is a Killing horizon of χ^μ. The idea of a Killing horizon is logically independent from that of an event horizons. However, in a spacetime with time-translational symmetry, a Killing horizon and an event horizon are closely related:

- Every event horizon Σ in a stationary, asymptotically flat spacetime is a Killing horizon for some Killing vector field χ^μ.
- If the spacetime is static, where it is invariant under time reversal $t \to -t$, the Killing vector field $\xi^\mu = (\partial_t)^\mu$ represents the time translation at infinity.
- If the spacetime is stationary but not static, it will have axial symmetry with a rotational Killing vector field $\eta^\mu = (\partial_\phi)^\mu$. Thus the Killing field related to the event horizon will be a linear combination $\chi^\mu = \xi^\mu + \alpha\eta^\mu$ for some constant α.

To every Killing horizon, we can associate a quantity called the surface gravity κ. The surface gravity of a black hole is the gravitational strength at the horizon measured by an observer at infinity. Surface gravity is constant anywhere on the surface of the black hole's horizon. Since χ^μ is a normal vector to Σ, it obeys the geodesic equation along the Killing horizon

$$\chi^\mu \nabla_\mu \chi^\nu = -\kappa \chi^\nu, \tag{2.63}$$

where the right-hand side is not zero since χ^μ does not have to be affinely parameterized. We can calculate the surface gravity associated to the Killing horizon Σ with the formula:

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu). \tag{2.64}$$

We will use this formula later after we verify that the constant α in χ^μ is actually the angular velocity of a particle at the horizon of the Kerr black hole. Since the Kerr solution is not static but stationary, the event horizons $r_\pm$ are not Killing horizons for $\xi = \partial_t$. The norm of ξ^μ is given by

$$\xi^\mu \xi_\mu = g_{tt} = -\left(1 - \frac{2Mr}{\Sigma}\right) = -\frac{\Delta - a^2 \sin^2\theta}{\Sigma}, \tag{2.65}$$

which does not vanish at the outer horizon. In fact, at $r = r_+$ where $\Delta = 0$, we have

$$\xi^\mu \xi_\mu = \frac{a^2 \sin^2\theta}{\Sigma} \geq 0. \tag{2.66}$$

This means that the Killing vector is already spacelike at the outer horizon, expect at the poles where $\theta = 0, \pi$ where it is null. The region between the Killing horizon for ξ^μ and the outer horizon r_+ is called the ergosphere. The ergosphere can be characterized by

$$\xi^\mu \xi_\mu = 0 \rightarrow (r - M)^2 = M^2 - a^2 \cos^2\theta, \tag{2.67}$$

and the outer event horizon is given by

$$\Delta(r_+) = 0 \rightarrow (r_+ - M)^2 = M^2 - a^2. \tag{2.68}$$

One interesting feature of a Kerr black hole is the ergosphere. Particles inside this region must move in the same direction as the black hole's rotation. In this region, they can either move towards the event horizon or move away and escape the ergosphere. To simply describe this, we use the angular velocity of a photon emitted in the ϕ direction at equatorial plane $\theta = \pi/2$, at a given radius r. The trajectory of a photon must be null, $ds^2 = 0$, so we get

$$0 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2, \tag{2.69}$$

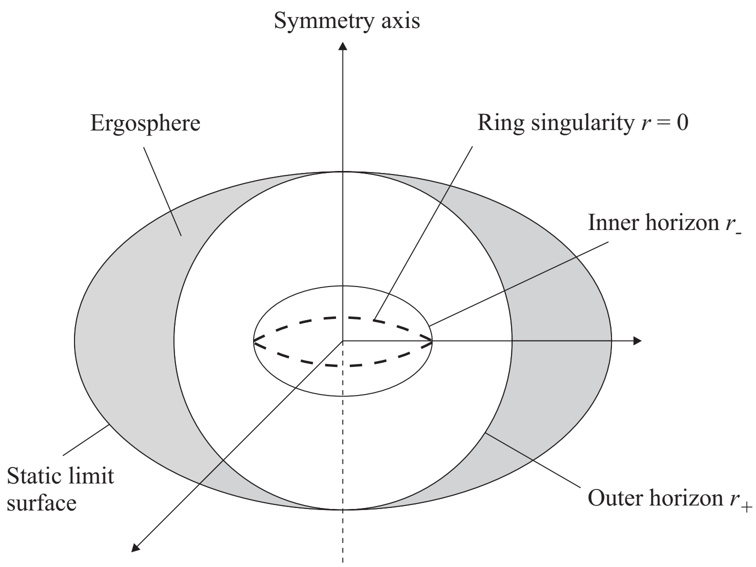


Figure 2.1: Illustration of a Kerr black hole.

which we can rewrite as

$$\frac{d\phi}{dt} = -\frac{g_{\tau\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}. \tag{2.70}$$

If we set $g_{tt} = 0$, we have two solutions:

$$\frac{d\phi}{dt} = 0, \quad \frac{d\phi}{dt} = -\frac{2g_{t\phi}}{g_{\phi\phi}} = \frac{a}{2M^2 + a^2}. \tag{2.71}$$

The first solution means the photon emitted against the black hole's rotation is at rest. Meanwhile the second solution means the photon moves around the black hole in the same direction as the black hole's rotation. The second solution gives $d\phi/dt$ of the same sign as the rotational parameter a. The angular velocity of a particle at the Kerr horizon can be calculated using Eq. (2.70):

$$\Omega_H = \frac{a}{2Mr_+} = \frac{a}{r_+^2 + a^2}. \tag{2.72}$$

The Killing vector associated to the event horizon is a linear combination of ξ^μ and η^μ:

$$\chi^\mu = \xi^\mu + \alpha\eta^\mu. \tag{2.73}$$

To find α, we examine that $\chi^\mu = 0$ at the horizon:

$$\chi^\mu\chi_\mu = 0 = \xi_\mu\xi^\mu + 2\alpha\eta_\mu\xi^\mu + \alpha^2\eta_\mu\eta^\mu = g_{tt} + 2\alpha g_{\phi t} + \alpha^2 g_{\phi\phi}. \tag{2.74}$$

Solving Eq. (2.74) for α, gives

$$\alpha = \frac{a}{{r_+}^2 + a^2}, \tag{2.75}$$

which exactly gives the angular velocity at the horizon, Ω_H. Now, we can write the linear combination (2.73) as

$$\chi^\mu = \xi^\mu + \Omega_H\eta^\mu. \tag{2.76}$$

Substituting Eq. (2.76) to the formula (2.64) we find the surface gravity of the Kerr event horizon:

$$\kappa = \frac{\sqrt{M^2 - a^2}}{2M(M + \sqrt{M^2 - a^2})} = \frac{r_+ - r_-}{4Mr_+}. \tag{2.77}$$

2.1.3 The law of thermodynamics and black holes mechanics

The analogue between a black hole's thermodynamics and the area of its event horizon was first noted by Jacob Bekenstein [16]. This is the first realization that black holes are thermodynamics systems. Hawking's theorem states that the area of a black hole's horizon never decreases [14], then Bekenstein proposed that the black hole entropy is proportional to the horizon area. A short time later, four laws of black hole mechanics were formulated [15]. Relations between some variables of black holes and thermodynamic systems are in the following Table 2.1.

Table 2.1: Thermodynamics and black holes.

Law	**Thermodynamics**	**Black holes mechanics**
Zeroth	T constant throughout body in thermal equilibrium.	κ constant over horizon of stationary black hole.
First	$dE = TdS + PdV +$ work terms.	$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ$.
Second	$\delta S \geq 0$ in any process.	$\delta A \geq 0$ in any process.
Third	Impossible to achieve $T = 0$ by a physical process.	Impossible to achieve $\kappa = 0$ by a physical process.

Briefly explained, in the zeroth law, the surface gravity of a black hole is uniform over an equilibrium (stationary) black hole. The surface gravity is analogue to the constant temperature for an equilibrium thermodynamic system. However, since general relativity is a classical theory, the temperature of a black hole is absolute zero. Thus, it would appear that the surface gravity could not represent a temperature. Nevertheless, Hawking discovered that a quantum particle creation effect results in particle emmision from a black hole with a blackbody spectrum at temperature:

$$T = \frac{\kappa}{2\pi}. \tag{2.78}$$

Thus, the surface gravity does represent the thermodynamic temperature of the black hole and the relation between thermodynamics and laws of black hole physics may be more than just an analogy. Next, the first law of mechanics for black holes reads

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ, \tag{2.79}$$

where J is the angular momentum of the black hole. Its respective thermodynamics conjugate is the angular velocity evaluated at the horizon. Using the correspondence with thermodynamics would appear to be straightforward:

$$dE = TdS + PdV + \text{work terms}. \tag{2.80}$$

Particularly, the term $\Omega_H dJ$ represents the thermodynamic work term PdV of the first law. The term dA appears in the same manner as dS appears in the first law of thermodynamics, expect that it is multiplied by $\kappa/8\pi$ rather than T. Next, the second law of black hole mechanics states that the area of the event horizon of a black hole never decreases in any physical process, which is in agreement with the second law of thermodynamics in which the total entropy of an isolated system can never decrease over time. Lastly, the third law of thermodynamics states that $S \to 0$ as $T \to 0$; This is not satisfied in black hole physics since the area of the horizon may remain finite as $\kappa \to 0$.

We now show that the area of the Kerr black hole horizon does not decrease in size. The area of the Kerr black hole is given in Eq. (2.54):

$$A = 8\pi M r_+. \tag{2.81}$$

Varying the area with respect to δM and δa gives

$$\begin{aligned}
\delta A &= 8\pi(r_+\delta M + M\delta r_+) \\
&= 8\pi\left[\left(M + \sqrt{M^2 - a^2}\right)\delta M + M\left(\delta M + \frac{M\delta M - a\delta a}{\sqrt{M^2 - a^2}}\right)\right] \\
&= \frac{8\pi}{\sqrt{M^2 - a^2}}\left[\sqrt{M^2 - a^2}\left(M + \sqrt{M^2 - a^2}\right)\delta M + \sqrt{M^2 - a^2}M\delta M + M^2\delta M - aM\delta a\right] \\
&= \frac{8\pi}{\sqrt{M^2 - a^2}}\left[2M\left(M + \sqrt{M^2 - a^2}\right)\delta M - a^2\delta M - aM\delta a\right] \\
&= \frac{8\pi}{\sqrt{M^2 - a^2}}(2Mr_+\delta M - a^2\delta M - aM\delta a).
\end{aligned} \tag{2.82}$$

From the definition of the rotational parameter $a = J/M$, we have the relation:

$$\delta J = a\delta M + M\delta a. \tag{2.83}$$

Using the expression above and the angular velocity (2.72) gives

$$\delta A = \frac{32\pi M r_+}{r_+ - r_-}(\delta M - \Omega_H \delta J), \tag{2.84}$$

or equivalently, the first law black hole mechanics:

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J. \tag{2.85}$$

Comparing with the first law of thermodynamics (2.80) we found the relations:

$$\text{Energy } E \Longleftrightarrow M \text{ Mass}, \tag{2.86}$$

$$\text{Temperature } T \Longleftrightarrow \frac{\kappa}{2\pi} \text{ Surface gravity}, \tag{2.87}$$

$$\text{Entropy } S \Longleftrightarrow \frac{A}{4} \text{ Horizon area}. \tag{2.88}$$

By substituting the surface gravity (2.77) and the horizon area (2.54), expressions (2.87) and (2.88) are famously known as the Hawking temperature formula and the Bekenstein-Hawking entropy formula for Kerr black holes, respectively:

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi} = \frac{r_+ - r_-}{8\pi M r_+}, \quad S_{\mathrm{BH}} = \frac{A}{4} = 2\pi M r_+. \tag{2.89}$$

2.2 Gravity with torsion

After general relativity was accepted as a new theory for gravitational fields, Weyl attempted to unify gravitation and electromagnetism in 1918. Unfortunately, his brilliant proposal did not succeed. However, he introduced the idea of gauge transformations and gauge invariance which are later considered as the foundations of gauge theory. Ten years later, Einstein made a similar attempt based on the mathematical structure of teleparallelism. The idea was introducing a field of orthonormal bases on the tangent spaces at each point of the four-dimensional spacetime called a tetrad. The tetrad has 16 components, while the spacetime tensor metric that represents the gravitational field has only 10 components. Einstein thought that the six additional degrees of freedom of the tetrad were related to six components of the electromagnetic field. However, his attempt of the unification did not succeed either. One of the reasons was the additional six degrees of freedom were eliminated by the six-parameter local Lorentz invariance of the theory. Later in 1920s, Cartan developed a modified theory of general relativity where the spacetime has curvature and torsion. The theory later came to be known as the Einstein-Cartan theory. Similar with Einstein's general relativity, in Einstein-Cartan theory, energy and momentum are the source of curvature, while a quantity, which later was discovered as spin, is the algebraic source of torsion. Interestingly, spin was not yet discovered when Cartan published his work.

After a few decades of hard work to develop a "new" general relativity by numerous people[2], Hayashi and Shirafuji combined all those ideas and made an approach where general relativity, a theory that involves only curvature, was supplemented by a teleparallel gravity, a theory that involves only torsion [51]. This new theory represented a new way of including torsion in general relativity, an alternative to the original proposal provided by Cartan in his theory. Today, this theory is known as the teleparallel equivalent of general relativity (TEGR) and its development is still yet to be explored. In TEGR, curvature and torsion provide equivalent description of gravity, but conceptual differences have to be considered. In GR, curvature gives the geometric picture of spacetime that describes the gravitational interaction. On the other hand, TEGR relates gravity with torsion. Torsion accounts for gravitational interaction by acting as a force. Over the last two decades, attempts to modify gravity in order to be able to describe the Universe's evolution as well as to reduce the non-renormalizable issue of GR have been made. Such modifications usually start by extending the Einstein-Hilbert action, i.e. replacing the Ricci scalar in (2.16) with an arbitrary function $f(R)$. However, it is also possible to modify the TEGR theory by replacing the torsion tensor with an arbitrary function $f(T)$. Such a theory is called $f(T)$ gravity. In this section, we briefly review the teleparallel gravity theory and its $f(T)$ extension, along with a rotating charged AdS black holes solution in quadratic $f(T)$ gravity based on Refs. [38, 50, 52, 53, 54, 55, 56].

[2] A detailed historical account of the teleparallel-based gravity theory can be found in Ref. [50]

2.2.1 Torsion tensor

We consider tetrad fields to describe the dynamics of spacetime, where they are defined at each point of the manifold as a base of orthonormal vectors e^A_μ. The capital Latin letters $A, B, C, \ldots$ denote the coordinates of the tangent spacetime (Minkowski spacetime) and the Greek letters $\mu, \nu, \rho, \ldots$ are the coordinates of the spacetime. The metric of the tetrad fields is $\eta_{AB} = \eta^{AB} = \mathrm{diag}(1, -1, -1, -1)$. Tetrad fields have the following properties:

$$e^\mu_A e^A_\nu = \delta^\mu_\nu, \quad e^\mu_A e^B_\mu = \delta^B_A. \tag{2.90}$$

Tetrad fields relate the curved spacetime metric $g_{\mu\nu}$ and the Minkowski metric η_{AB} as

$$g_{\mu\nu} = \eta_{AB} e^A_\mu e^B_\nu, \quad g^{\mu\nu} = \eta^{AB} e^\mu_A e^\nu_B. \tag{2.91}$$

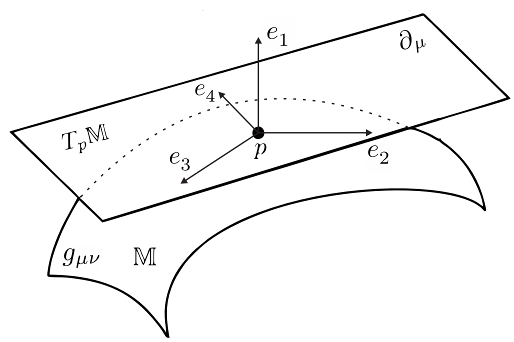


Figure 2.2: Four-dimensional spacetime manifold $\mathbb{M}$ with a metric $g_{\mu\nu}$ where tetrad fields $e_A = e^\mu_A \partial_\mu$ are defined at point p on the tangent space T_P.

In the construction of the torsion tensor, it would be useful to introduce bivectors $B^{\mu\nu}$ which are obtained by anti-symmetric product of two vectors where they satisfy the relation[3]

$$B^{[\mu\nu} B^{\alpha]\beta} = 0. \tag{2.92}$$

The bivector $B_{\mu\nu}$ takes the form

$$B^{\mu\nu} = B^{AB} e^\mu_A e^\nu_B, \tag{2.93}$$

with $B^{AB} = -B^{BA}$. We define the torsion tensor from the anti-symmetric part of the affine connection as

$$T^\alpha_{\mu\nu} = \frac{1}{2}\left(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}\right) \equiv \Gamma^\alpha_{[\mu\nu]}. \tag{2.94}$$

Note that $T^\alpha_{\mu\nu}$ is zero in GR. In calculations, sometimes torsion occurs in linear combination as in the contortion tensor, defined as

$$K^\alpha_{\mu\nu} = -T^\alpha_{\mu\nu} - T^\alpha_{\mu\nu} + T^\alpha_{\nu\mu}, \tag{2.95}$$

[3] A shorthand notation for anti-symmetric tensor is denoted by a pair of square brackets.

where $T_\mu \equiv T^\nu_{\mu\nu}$. From this definition, we can write the affine connection as

$$\Gamma^\alpha_{\mu\nu} = \left\{ {}^{\alpha}_{\mu\nu} \right\} - K^\alpha_{\mu\nu}, \tag{2.96}$$

where $\left\{ {}^{\alpha}_{\mu\nu} \right\}$ are the Christoffel symbols of the symmetric Levi-Civita connection. The non-zero torsion in the affine connection suggests that the the covariant derivatives of a scalar field φ do not commute:

$$\tilde{\nabla}_{[\mu}\tilde{\nabla}_{\nu]}\varphi = -T^\alpha_{\mu\nu}\tilde{\nabla}_\alpha\varphi, \tag{2.97}$$

and the relation between a vector u^μ and a covector v_μ are given as

$$\left(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \tilde{\nabla}_\nu\tilde{\nabla}_\mu\right)u^\alpha = R^\alpha_{\mu\nu\beta}u^\beta - 2T^\beta_{\mu\nu}\tilde{\nabla}_\beta u^\alpha, \quad \left(\tilde{\nabla}_\mu\tilde{\nabla}_\nu - \tilde{\nabla}_\nu\tilde{\nabla}_\mu\right)v_\beta = R^\beta_{\mu\nu\alpha}v_\beta - 2T^\beta_{\mu\nu}\tilde{\nabla}_\beta v_\alpha, \tag{2.98}$$

where the Riemann tensor is defined as

$$R^\beta_{\mu\nu\alpha} = \partial_\mu\Gamma^\beta_{\nu\alpha} - \partial_\nu\Gamma^\beta_{\mu\alpha} + \Gamma^\beta_{\mu\lambda}\Gamma^\lambda_{\nu\alpha} - \Gamma^\beta_{\nu\lambda}\Gamma^\lambda_{\mu\alpha}. \tag{2.99}$$

Torsion plays a part to the Riemann tensor, the Ricci tensor, and the Ricci scalar which can be explicitly given as

$$R^\beta_{\mu\nu\alpha} = R^\beta_{\mu\nu\alpha}\left(\{\}\right) - \nabla_\mu K^\beta_{\nu\alpha} + \nabla_\nu K^\beta_{\mu\alpha} + K^\nu_{\mu\lambda}K^\lambda_{\nu\alpha} - K^\beta_{\nu\lambda}K^\lambda_{\nu\alpha}, \tag{2.100}$$

$$R_{\mu\nu} = R_{\mu\nu}\left(\{\}\right) - 2\nabla_\mu T_\alpha + \nabla_\nu K^\nu_{\mu\alpha} + K^\nu_{\mu\lambda}K^\lambda_{\nu\alpha} - 2T_\lambda K^\lambda_{\mu\alpha}, \tag{2.101}$$

$$R = R\left(\{\}\right) - 4\nabla_\mu T^\mu + K_{\alpha\lambda\nu}K^{\nu\alpha\lambda} - 4T_\mu T^\mu. \tag{2.102}$$

Here, $R^\beta_{\mu\nu\alpha}\left(\{\}\right)$, $R_{\mu\nu}\left(\{\}\right)$, and $R\left(\{\}\right)$ are Riemann tensor, Ricci tensor, and Ricci scalar, respectively, of the symmetric connection. Also, $\tilde{\nabla}$ and ∇ are the covariant derivative with and without torsion, respectively.

2.2.2 Teleparallel equivalent of general relativity and the $f(T)$ extension

In the TEGR theory, we use the same notation as the previous section in which the capital Latin letters denote the coordinates of the tangent spacetime and the Greek letters are the coordinates of the spacetime. The metric of the tangent space is Minkowskian in which $\eta_{AB} = \eta^{AB} = \mathrm{diag}(1,-1,-1,-1)$. First, we introduce the gauge transformation

$$x'^A = x^A + \alpha^A, \tag{2.103}$$

where $\alpha^A \equiv \alpha^A\left(x^\mu\right)$ are the transformation parameters. We also define the infinitesimal translational generators $P_A = \frac{\partial}{\partial x^A} \equiv \partial_A$ which are the differential operators. These generators satisfy the commutation relation $[P_A, P_B] = 0$. The corresponding infinitesimal transformation can be written as

$$\delta x^A = \delta\alpha^B P_B x^A. \tag{2.104}$$

We also define the gauge covariant derivative for a matter field ψ as

$$D_\mu\psi = e^B_\mu\partial_B\psi, \tag{2.105}$$

where the non-trivial tetrad field is

$$e_\mu^B = \partial_\mu x^B + A_\mu^B, \tag{2.106}$$

with A_μ^B being the gauge potentials. As it is standard in Abelian gauge theory, the field strength tensor is defined as

$$F_{\mu\nu}^B = \partial_\mu A_\nu^B - \partial_\nu A_\mu^B, \tag{2.107}$$

which satisfies the commutation relation

$$[D_\mu, D_\nu]\,\psi = F_{\mu\nu}^B P_B \psi. \tag{2.108}$$

It is worth mentioning that, while the tangent spacetime indices are raised and lowered by the metric η_{AB}, the spacetime indices are raised and lowered by the Riemannian metric $g_{\mu\nu} = \eta_{AB} e_\mu^A e_\nu^B$.

In TEGR, a non-trivial tetrad field induces on spacetime a teleparallel structure directly related to the gravitational field. This means that given a non-trivial tetrad, it is possible to define the Weitzenbock connection

$$W_{\mu\nu}^\alpha = e_A^\alpha \partial_\nu e_\mu^A = -e_\mu^A \partial_\nu e_A^\alpha, \tag{2.109}$$

which is curvature-free and characterized by torsion. Hence, the Weitzenbock covariant derivative of the tetrad field is

$$\nabla_\nu e_\mu^A = \partial_\nu e_\mu^A - W_{\mu\nu}^\alpha e_\alpha^A = 0. \tag{2.110}$$

Furthermore, we can express the torsion tensor as

$$T_{\mu\nu}^\alpha = W_{\nu\mu}^\alpha - W_{\mu\nu}^\alpha, \tag{2.111}$$

from which the gravitational *force* results from the torsion written in the tetrad basis, which is

$$F_{\mu\nu}^A = e_\alpha^A T_{\mu\nu}^\alpha. \tag{2.112}$$

We can also define the torsionless Levi-Civita symmetric connection using the non-trivial tetrad field as

$$\tilde{\Gamma}_{\mu\nu}^\beta = \frac{1}{2} g^{\beta\alpha} \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right). \tag{2.113}$$

The contorsion tensor

$$K_{\mu\nu}^\alpha = \frac{1}{2}\left(T_{\mu\nu}^\rho + T_{\nu\mu}^\rho - T_{\mu\nu}^\rho\right), \tag{2.114}$$

relates the Weitzenbock connections and the Levi-Civita connections as

$$W_{\mu\nu}^\alpha = \tilde{\Gamma}_{\mu\nu}^\alpha + K_{\mu\nu}^\alpha. \tag{2.115}$$

From the Riemann tensor identity

$$R_{\lambda\mu\nu}^\alpha = \partial_\mu W_{\lambda\nu}^\alpha - \partial_\nu W_{\lambda\mu}^\alpha + W_{\beta\mu}^\alpha W_{\lambda\nu}^\beta - W_{\beta\nu}^\alpha W_{\lambda\mu}^\beta \equiv 0, \tag{2.116}$$

substituting Eq. (2.115), we get

$$R^{\alpha}_{\lambda\mu\nu} = \tilde{R}^{\alpha}_{\lambda\mu\nu} + U^{\alpha}_{\lambda\mu\nu} \equiv 0, \tag{2.117}$$

with

$$U^{\alpha}_{\lambda\mu\nu} = D_\mu K^{\alpha}_{\lambda\nu} - D_\nu K^{\alpha}_{\lambda\mu} + K^{\beta}_{\lambda\nu} K^{\alpha}_{\beta\mu} - K^{\beta}_{\lambda\mu} K^{\alpha}_{\beta\nu}, \tag{2.118}$$

where $\tilde{R}^{\alpha}_{\lambda\mu\nu}$ is the Riemann tensor of the Levi-Civita connection and $U^{\alpha}_{\lambda\mu\nu}$ is a tensor expressed in terms of the Weitzenbock connection, and D_μ is the teleparallel covariant derivative. We can see the explicit form of the teleparallel covariant derivative by acting on a vector V^μ:

$$D_\alpha V^\mu \equiv \partial_\alpha V^\mu + \left(W^{\mu}_{\lambda\alpha} - K^{\mu}_{\lambda\alpha}\right) V^\lambda. \tag{2.119}$$

From identity (2.117), we can see the equivalence between teleparallel and Riemann gravitational interaction. The Riemann tensor from the Levi-Civita connection $\tilde{R}^{\alpha}_{\lambda\mu\nu}$ compensates the one from the Weitzenbock connection $U^{\alpha}_{\lambda\mu\nu}$. Thus $R^{\alpha}_{\lambda\mu\nu}$ is equal zero.

The gravitational field Lagrangian of TG can be written in terms of a quadratic torsion tensor. Thus we consider writing the action of a gravitational field in TG as

$$\mathcal{S}_{\mathrm{G}} = \frac{1}{16\pi} \int d^4x e S^{\alpha\mu\nu} T_{\alpha\mu\nu}, \tag{2.120}$$

where $e = \det\left(e^{A}_{\mu}\right)$, and the superpotential tensor $S^{\alpha\mu\nu}$ in terms of the Weitzenbock connection is

$$S^{\alpha\mu\nu} = -S^{\alpha\nu\mu} \equiv \frac{1}{2}\left(K^{\mu\nu\alpha} - g^{\alpha\nu} T^{\beta\mu}_{\beta} + g^{\alpha\mu} T^{\beta\nu}_{\beta}\right). \tag{2.121}$$

Therefore, the gravitational field Lagrangian can be written as

$$\mathcal{L}_{\mathrm{G}} = \frac{e}{16\pi} T_{\alpha\mu\nu} S^{\alpha\mu\nu}, \tag{2.122}$$

The Lagrangian (2.122) can be expressed in terms of a quadratic torsion tensor by substituting the superpotensial tensor (2.121) and making use of the identity $T^{\mu}_{\mu\rho} = K^{\mu}_{\rho\mu}$:

$$\begin{aligned}
\mathcal{L}_{\mathrm{G}} &= \frac{e}{16\pi} T_{\alpha\mu\nu} S^{\alpha\mu\nu} \\
&= \frac{e}{32\pi} T_{\alpha\mu\nu}\left(K^{\mu\nu\alpha} - g^{\alpha\nu} T^{\beta\mu}_{\beta} + g^{\alpha\mu} T^{\beta\nu}_{\beta}\right) \\
&= \frac{e}{32\pi}\left(T_{\alpha\mu\nu} K^{\mu\nu\alpha} - T_{\alpha\mu\nu} g^{\alpha\nu} T^{\beta\mu}_{\beta} + T_{\alpha\mu\nu} g^{\alpha\mu} T^{\beta\nu}_{\beta}\right) \\
&= \frac{e}{32\pi}\left[\frac{1}{2} T_{\alpha\mu\nu}\left(T^{\nu\mu\alpha} + T^{\alpha\mu\nu} - T^{\mu\nu\alpha}\right) - T^{\alpha}_{\mu\nu} T^{\beta\mu}_{\beta} + T^{\mu}_{\mu\nu} T^{\beta\nu}_{\beta}\right] \\
&= \frac{e}{32\pi}\left(\frac{1}{2} T_{\alpha\mu\nu} T^{\nu\mu\alpha} + \frac{1}{2} T_{\alpha\mu\nu} T^{\alpha\mu\nu} - \frac{1}{2} T_{\alpha\mu\nu} T^{\mu\nu\alpha} - T^{\alpha}_{\mu\nu} T^{\beta\mu}_{\beta} - T^{\mu}_{\nu\mu} T^{\beta\nu}_{\beta}\right) \\
&= \frac{e}{16\pi}\left(\frac{1}{4} T_{\alpha\mu\nu} T^{\alpha\mu\nu} + \frac{1}{2} T_{\alpha\nu\mu} T^{\mu\nu\alpha} - T^{\mu}_{\nu\mu} T^{\beta\nu}_{\beta}\right).
\end{aligned} \tag{2.123}$$

The first term of the Lagrangian corresponds to the Lagrangian of electromagnetism. The existence of the tangent space resulting the combination of a torsion tensor contraction in the other two terms.

Now, consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathrm{matter}}, \tag{2.124}$$

where $\mathcal{L}_{\text{matter}}$ is the general matter field Lagrangian. Varying the Lagrangian (2.124) with respect to the gauge potential A^B_ρ, or equivalently, with respect to the tetrad field e^B_μ leads us to the teleparallel version of the Einstein gravitational field equation

$$\partial_\sigma(eS_B^{\rho\sigma}) - 8\pi e j_b^\rho = 8\pi e \mathcal{T}_{\text{matter}}{}^\rho_B. \tag{2.125}$$

In the field equation, the superpotential is represented by

$$eS_B^{\rho\sigma} = -8\pi \frac{\partial \mathcal{L}_{\text{G}}}{\partial(\partial_\sigma e^B_\rho)}, \tag{2.126}$$

and the gauge current is

$$ej_b^\rho = -\frac{\partial \mathcal{L}_{\text{G}}}{\partial e^B_\rho}, \tag{2.127}$$

where it represents the Noether energy-momentum density of gravitation itself. The energy-momentum tensor is then

$$e\mathcal{T}_{\text{matter}}{}^\rho_B = -\frac{\delta \mathcal{L}_{\text{matter}}}{\delta e^B_\rho} \equiv -\left(\frac{\partial \mathcal{L}_{\text{matter}}}{\partial e^B_\rho} - \partial_\mu \frac{\partial \mathcal{L}_{\text{matter}}}{\partial_\mu \partial e^B_\rho}\right). \tag{2.128}$$

The superpotential (2.126) and the energy-momentum current (2.127) are found to be

$$S_B^{\rho\sigma} = K_B^{\rho\sigma} - e^\sigma_B T_\theta^{\theta\rho} + e^\rho_B T_\theta^{\theta\sigma}, \tag{2.129}$$

and

$$j_B^\rho = \frac{1}{8\pi} e^\mu_B S_C^{\nu\rho} T^C_{\nu\mu} - \frac{e^\rho_B}{e}\mathcal{L}_{\text{matter}} + \frac{1}{8\pi} B^C_{B\sigma} S_C^{\rho\sigma}, \tag{2.130}$$

respectively, where $B^C_{B\sigma} = \Lambda^C_D \partial_\sigma \Lambda^D_B$ is called the spin connection of teleparallel gravity and Λ^C_D is the Lorentz transformation matrix. In the sense of special relativity, it represents only the inertial properties of the frame, not gravitation. In addition, using the spin connection, we can construct the torsion $T^A_{\sigma\rho} = \partial_\sigma e^A_\rho - \partial_\rho e^A_\sigma + B^A_{C\sigma} e^C_\rho + B^A_{C\rho} e^C_\sigma$. Due to the anti-symmetric property of the superpotential in the last two indices, the gravitational plus the energy-momentum density is conserved in the ordinary sense:

$$\partial_\rho(ej_B^\rho + e\mathcal{T}_{\text{matter}}{}^\rho_B) = 0. \tag{2.131}$$

Using the identity (2.115), the left-hand side of field equation (2.125) is equal to

$$\partial_\sigma(eS_B^{\rho\sigma}) - 8\pi e j_b^\rho = e\left(\tilde{R}^\rho_B - \frac{1}{2} e^\rho_B \tilde{R}\right). \tag{2.132}$$

This means that the teleparallel field equations (2.125) are equivalent to the Einstein's field equations

$$\tilde{R}^\rho_B - \frac{1}{2} e^\rho_B \tilde{R} = 8\pi \mathcal{T}_{\text{matter}}{}^\rho_B. \tag{2.133}$$

It is worth mentioning that general relativity has both metric and tetrad formulation. In the metric formalism, we calculate the Riemann curvature tensor using Christoffel symbols from the metric tensor. Thus the dynamics of the metric tensor is described by the Einstein field equations

$$\tilde{R}^\rho_\sigma - \frac{1}{2} \delta^\rho_\sigma \tilde{R} = 8\pi \mathcal{T}_{\text{matter}}{}^\rho_\sigma. \tag{2.134}$$

In tetrad formalism, the 10-components of the metric tensor are replaced by the 16-components of the tetrad field. Thus, the Einstein field equation takes the form (2.133), where the Ricci tensor is calculated directly from the tetrad and is related to the spacetime-only-indices Ricci tensor by $R^A_\nu = e^A_\mu R^\mu_\nu$.

From this result, we have observed that the tetrad form of the Einstein equations are just a projection of their spacetime form along the tetrad components. Hence, the dynamical content of both tetrad and spacetime forms is equivalent and they determine only the tensor metric. In addition, the teleparallel Einstein equations (2.133) are covariant under local Lorentz transformation:

$$\Lambda^B_C(x)\left(\tilde{R}^\rho_B - \frac{1}{2}e^\rho_B\tilde{R}\right) = 8\pi\Lambda^B_C(x)\mathcal{T}_{\text{matter}}{}^\rho_B. \tag{2.135}$$

As we mentioned before, in the beginning of chapter 2.2, this covariance eliminates six of the 16 equations (2.133), which means that the tetrad is determined by (2.133) only up to a local Lorentz transformation. In other words, we only determine the metric tensor. We can expect this because both metric and tetrad formalisms are two equivalent formalisms of the same theory.

As well as in the TEGR, the dynamic variable of the $f(T)$ gravity is the tetrad field e_A. We also use the same notation in which Greek letters denote spacetime coordinates and capital Latin letters denote tangent space coordinates. In $f(T)$ gravity, we also use the Weitzenbock connection (2.109) which is curvature-free. Thus, it has a non-vanishing torsion $T^\alpha_{\mu\nu} = W^\alpha_{\nu\mu} - W^\alpha_{\mu\nu} = e^\alpha_A\left(\partial_\mu e^A_\nu - \partial_\nu e^A_\mu\right)$. We define the torsion scalar by

$$T = T^\alpha_{\mu\nu}S^{\mu\nu}_\alpha, \tag{2.136}$$

where the superpotential tensor is $S^{\mu\nu}_\alpha = \frac{1}{2}\left(K^{\mu\nu}_\alpha + \delta^\mu_\alpha T^{\beta\nu}_\beta - \delta^\nu_\alpha T^{\beta\mu}_\beta\right)$. We note that the contortion tensor $K_{\alpha\mu\nu}$ is given by $K_{\alpha\mu\nu} = \frac{1}{2}\left(T_{\nu\alpha\mu} + T_{\alpha\mu\nu} - T_{\mu\alpha\nu}\right)$.

In teleparallel gravity, the action of the theory is constructed by the torsion scalar or the the teleparallel Lagrangian density T. Along the same lines as the generalization of the Ricci scalar to an arbitrary function $f(R)$ in the Einstein-Hilbert action, we generalized the torsion scalar T to an arbitrary function $f(T)$. Specifically, the action in a Universe dictated by $f(T)$ gravity reads:

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\,|e|\,f(T) + \int d^4x\,|e|\,\mathcal{L}_{\text{matter}}, \tag{2.137}$$

where $|e| = \det\left(e^A_\mu\right) = \sqrt{-g}$, and $\mathcal{L}_{\text{matter}}$ is the matter Lagrangian. In the case of $f(T) = T$, the $f(T)$ gravity becomes TEGR.

Let us consider varying the action (2.137) with respect to the tetrad and obtain the field equation:

$$e^\rho_A S^{\mu\nu}_\rho(\partial_\mu T)f''(T) + \left[e^{-1}\partial_\mu(ee^\rho_A S^{\mu\nu}_\rho) - e^\lambda_A T^\rho_{\mu\lambda}S^{\nu\mu}_\rho\right]f'(T) + \frac{1}{4}e^\nu_A f(T) = -4\pi e^\rho_A\mathcal{T}_{\text{matter}}{}^\nu_\rho. \tag{2.138}$$

Multiplying (2.138) by e^A_μ, using the identity (2.90), and rearranging some indices, we get

$$S^{\rho\nu}_\mu(\partial_\rho T)f''(T) + \left[e^{-1}e^A_\mu\partial_\rho(ee^\lambda_A S^{\rho\nu}_\lambda) - T^\rho_{\lambda\mu}S^{\nu\lambda}_\rho\right]f'(T) + \frac{1}{4}\delta^\nu_\mu f(T) = -4\pi\mathcal{T}_{\text{matter}}{}^\nu_\mu, \tag{2.139}$$

where $f'(T)$ and $f''(T)$ denote the first and second order derivatives of $f(T)$ with respect to the torsion scalar, and $\mathcal{T}_{\text{matter}}{}^\nu_\mu$ is the energy momentum tensor which is constructed by the matter Lagrangian, $\mathcal{L}_{\text{matter}}$.

2.2.3 Black holes in quadratic $f(T)$ gravity

We consider a four-dimensional rotating charged AdS black hole solution in $f(T)$-Maxwell theory with a negative cosmological constant where

$$f(T) = T + \alpha T^2. \tag{2.140}$$

The dimensional negative parameter α is the coefficient of the quadratic term of the scalar torsion. The action of the $f(T)$-Maxwell theory in 4D, for an asymptotically AdS spacetimes, is given by

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\, |e|\, (f(T) - 2\Lambda) + \int d^4x\, |e|\, \mathcal{L}_{\rm em}, \tag{2.141}$$

where $\Lambda = -3/l^2$ is the 4D cosmological constant, l is the length scale of AdS spacetime, and $\mathcal{L}_{\rm em} = -\frac{1}{2}F^{\mu\nu}F_{\mu\nu}$ is the Maxwell Lagrangian. In action (2.141), $F = d\tilde{\Phi}$, where $\tilde{\Phi} = \tilde{\Phi}_\mu dx^\mu$ is the gauge potential one-form.

Varying action (2.141) with respect to the tetrad fields and the Maxwell potential Φ_μ, one finds the field equations for gravity

$$H^\nu_\mu \equiv S^{\rho\nu}_\mu(\partial_\rho T)f''(T) + \left[e^{-1}e^A_\mu \partial_\rho(ee^\lambda_A S^{\rho\nu}_\lambda) - T^\rho_{\lambda\mu}S^{\nu\lambda}_\rho\right] f'(T) + \frac{1}{4}\delta^\nu_\mu(f(T) - 2\Lambda) = -4\pi {\mathcal{T}_{\rm em}}^\nu_\mu, \tag{2.142}$$

and Maxwell's equations

$$\partial_\nu\left(\sqrt{-g}F^{\mu\nu}\right) = 0, \tag{2.143}$$

respectively. In Eq. (2.142), ${\mathcal{T}_{\rm em}}^\nu_\mu = F_{\mu\alpha}F^{\nu\alpha} - \frac{1}{4}\delta^\nu_\mu F_{\alpha\beta}F^{\alpha\beta}$, is the energy-momentum tensor of the electromagnetic field.

Asymptotically AdS solution

We implement the $f(T)$ gravity field equation (2.142) into the flat 4D spacetime horizon in cylindrical coordinates (t, r, ϕ, z), in which the tetrad fields read as

$$e^\mu_1 = \left(\sqrt{A(r)}, 0, 0, 0\right), \quad e^\mu_2 = \left(0, \frac{1}{\sqrt{B(r)}}, 0, 0\right), \quad e^\mu_3 = (0, 0, r, 0)\,, \quad e^\mu_4 = (0, 0, 0, r)\,, \tag{2.144}$$

where $A(r)$ and $B(r)$ are unknown functions of the radial coordinate r and the range of coordinates are given by $-\infty < t < \infty$, $-\infty < z < \infty$, $0 \le r < \infty$, and $0 \le \phi < 2\pi$. We calculate the torsion scalar by substituting the tetrad fields to Eq. (2.136):

$$T = \frac{4A'B}{rA} + \frac{2B}{r^2}. \tag{2.145}$$

In the vacuum case, ${\mathcal{T}_{\rm matter}}^\nu_\mu = 0$, and the non-vanishing components of field equations (2.139) read

$$H^t_t = \frac{4Bf''(T)T'}{r} + \frac{2f'(T)}{r^2A}(2AB + rBA' + rAB') - f(T) + 2\Lambda = 0, \tag{2.146}$$

$$H^r_r = 2Tf'(T) + 2\Lambda - f(T) = 0, \tag{2.147}$$

$$H_\phi^\phi = H_z^z = \frac{f''(T)\left(r^2T+2B\right)T'}{2r} + \frac{f'(T)}{2r^2A^2}\big[2r^2ABA'' - r^2BA'^2 + 6rABA' + r^2AA'B' \\ +2A^2\left(2B+rB'\right)\big] - f(T) + 2\Lambda = 0. \quad (2.148)$$

For the specific form of $f(T) = T + \alpha T^2$, the above differential equations can be written as

$$H_t^t = \frac{8B\alpha T'}{r} + \frac{2(1+2\alpha T)}{r^2A}\left(2AB + rBA' + rAB'\right) - T - \alpha T^2 + 2\Lambda = 0, \quad (2.149)$$

$$H_r^r = T + 3\alpha T^2 + 2\Lambda = 0, \quad (2.150)$$

$$H_\phi^\phi = H^z{}_z = \frac{2\alpha\left(r^2T+2B\right)T'}{2r} + \frac{(1+2\alpha T)}{2r^2A^2}\big[2r^2ABA'' - r^2BA'^2 + 6rABA' + r^2AA'B' \\ +2A^2\left(2B+rB'\right)\big] - T - \alpha T^2 + 2\Lambda = 0. \quad (2.151)$$

In the above solutions, we use a notation in which $A(r) = A$, $B(r) = B$, $A' \equiv \frac{dA}{dr}$, and $B' \equiv \frac{dB}{dr}$ for simplicity. The solution of differential equations (2.149)–(2.151) is

$$A(r) = -\frac{r^2}{36\alpha} - \frac{M}{r}, \quad B(r) = A(r), \quad (2.152)$$

where M is the mass parameter.

Charged AdS solution

Next, we use the 4D spacetime of tetrad fields (2.144) with a Maxwell potential $\Phi^\mu = (\Phi(r), 0, 0, 0)$. Thus, the non-zero components of the field equations are the following:

$$H_t^t = \frac{4Bf''(T)T'}{r} + \frac{2f'(T)}{r^2A}\left(2AB + rBA' + rAB'\right) - f(T) + 2\Lambda + \frac{2\Phi'^2(r)B}{A} = 0, \quad (2.153)$$

$$H_r^r = 2Tf'(T) + 2\Lambda - f(T) + \frac{2\Phi'^2(r)B}{A} = 0, \quad (2.154)$$

$$H_\phi^\phi = H_z^z = \frac{f''(T)\left(r^2T+2B\right)T'}{2r} + \frac{f'(T)}{2r^2A^2}\big[2r^2ABA'' - r^2BA'^2 + 4A^2B + 6rABA' \\ +r^2AA'B' + 2rA^2B'\big] - f(T) + 2\Lambda - \frac{2\Phi'^2(r)B}{A} = 0, \quad (2.155)$$

where $\Phi' = \frac{d\Phi}{dr}$. The 4D solution of the above differential equations is

$$A(r) = \frac{r^2{c_2}^4}{54{c_3}^2} + \frac{c_1}{r} + \frac{3{c_2}^2}{2r^2} + \frac{2c_2c_3}{2r^4}, \quad B(r) = A(r)\beta(r), \quad (2.156)$$

where

$$\beta(r) = -\frac{9{c_3}^2\left(1+\frac{3c_3}{c_2r^2}\right)^2}{6\alpha{c_2}^4}, \quad \Phi(r) = \frac{c_2}{r} + \frac{c_3}{r^3}. \quad (2.157)$$

To obtain an asymptotically AdS solution we set

$${c_3}^2 = -\frac{6{c_2}^4\alpha}{9}, \quad (2.158)$$

where we notice that α should be negative, or else the situation is non-physical. Accordingly, the monopole momentum is related to the quadruple momentum of the solution. In such a case, we get

$$A(r) = \frac{r^2}{36\,|\alpha|} - \frac{M}{r} + \frac{3Q^2}{2r^2} + \frac{2Q^3\sqrt{6\,|\alpha|}}{6r^4}, \quad B(r) = A(r)\beta(r), \quad \beta(r) = \left(1 + \frac{Q\sqrt{6\,|\alpha|}}{r^2}\right)^2,$$

$$\Phi(r) = \frac{Q}{r} + \frac{Q^2\sqrt{6\,|\alpha|}}{3r^3}, \tag{2.159}$$

where we set $c_1 = -M$, $c_2 = Q$ is the monopole momentum, and $\frac{Q^2\sqrt{6|\alpha|}}{3}$ is the quadruple momentum. The spacetime that can be generated by the tetrad fields (2.144) reads

$$ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2\left(d\phi^2 + \frac{dz^2}{l^2}\right). \tag{2.160}$$

We should note that in taking the limit $\alpha \to 0$, this theory does not have a correspondence with the TEGR solution. In other words, this charged spacetime solution has no analogue in GR. In addition, taking the limit $Q \to 0$, we recover the AdS neutral black hole solution (2.152). Finally, we would like to note that the rotating black hole solution in $f(T)$ gravity and its thermodynamics aspects will be discussed later in chapter 5.

3 Introducing Conformal Field Theory

Conformal field theory is one of the most important tools in theoretical high energy physics. It has many applications outside of string theory, particularly in statistical mechanics where it offers a description of critical behaviour in a system. In the last decades, notably in gravitational high energy physics, attention has been concentrated on CFT due to its role in the AdS/CFT correspondence conjecture. A conformal transformation is a smooth change of coordinates x^μ to x'^μ such that the line element changes as

$$g_{\mu\nu}(x) = \Omega^2(x)\, g_{\mu\nu}(x)\,, \tag{3.1}$$

where $\Omega(x)$ is an arbitrary real function and $g_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$ is the metric tensor. A CFT is a QFT which is invariant under conformal transformation (3.1). The physics of the field theory is the same at all length scales. A CFT preserves angles between two vectors but not distances. The conformal transformation (3.1) has a different meaning depending on the metric tensor $g_{\mu\nu}$. The fluctuating background metric makes the conformal transformation a diffeomorphism, in other words, a gauge symmetry. The constant background makes it a physical symmetry, taking the coordinate x^μ to x'^μ.

In this thesis, we review some basic concepts in CFT and focus more on 2D CFT. First, we will discuss the conformal group in higher dimensions and then conformal group in 2D. Second, we will discuss the Witt algebra and its extension: the Virasoro algebra where we introduce the central charge. Third, we will briefly discuss the primary field after considering the energy-momentum tensor and Noether currents. In the last section, we will derive the Cardy entropy formula of 2D CFT. The main references for this chapter are Refs. [41, 46, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70].

3.1 Conformal group in $N > 2$ dimensions

The infinitesimal generators of the conformal group can be specified by considering the infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$. The corresponding change in the metric tensor $g_{\mu\nu}$ to first order in ϵ is

$${g'}_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = \left(\delta^\alpha_\mu - \delta_\mu \epsilon^\alpha\right)\left(\delta^\beta_\nu - \delta_\nu \epsilon^\beta\right) g_{\alpha\beta} = g_{\mu\nu} - \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) + \mathcal{O}\left(\epsilon^2\right). \tag{3.2}$$

To satisfy (3.1), we require that $(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \propto \eta_{\mu\nu}$, thus

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x)\, \eta_{\mu\nu}. \tag{3.3}$$

To determine the factor $f(x)$, we take the trace on both sides of Eq. (3.3):

$$\begin{aligned}
\eta^{\mu\nu}\partial_\mu\epsilon_\nu + \eta^{\mu\nu}\partial_\nu\epsilon_\mu &= f(x)\,\eta_{\mu\nu}\eta^{\mu\nu} \\
\partial^\nu\epsilon_\nu + \partial^\mu\epsilon_\mu &= f(x)\,N \\
2\partial^\rho\epsilon_\rho &= f(x)\,N \\
f(x) &= \frac{2}{N}(\partial\cdot\epsilon)\,, \qquad (3.4)
\end{aligned}$$

then substitute the result back to Eq. (3.3):

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{N}(\partial\cdot\epsilon)\,\eta_{\mu\nu}. \qquad (3.5)$$

This equation is called the conformal Killing equation and ϵ^μ is called the conformal Killing vector. Comparing Eq. (3.5) to Eq. (3.1), we find the arbitrary function $\Omega(x) = 1+(2/N)(\partial\cdot\epsilon)$. By applying an extra derivative ∂^ν on Eq. (3.5), we find

$$\begin{aligned}
\partial_\mu\partial^\nu\epsilon_\nu + \partial^\nu\partial_\nu\epsilon_\mu &= \frac{2}{N}\partial_\mu(\partial\cdot\epsilon) \\
\partial_\mu(\partial\cdot\epsilon) + \Box\epsilon_\mu &= \frac{2}{N}\partial_\mu(\partial\cdot\epsilon)\,. \qquad (3.6)
\end{aligned}$$

Again, applying an extra derivative ∂_ν on Eq. (3.6), we find

$$\partial_\nu\partial_\mu(\partial\cdot\epsilon) + \Box\partial_\nu\epsilon_\mu = \frac{2}{N}\partial_\nu\partial_\mu(\partial\cdot\epsilon)\,, \qquad (3.7)$$

and when we interchange μ with ν, we find

$$\partial_\mu\partial_\nu(\partial\cdot\epsilon) + \Box\partial_\mu\epsilon_\nu = \frac{2}{N}\partial_\mu\partial_\nu(\partial\cdot\epsilon)\,. \qquad (3.8)$$

Finally, summing Eq. (3.7) and Eq. (3.8), we end up with the deformed conformal Killing equation

$$\begin{aligned}
(\partial_\nu\partial_\mu + \partial_\mu\partial_\nu)(\partial\cdot\epsilon) + \Box(\partial_\nu\epsilon_\mu + \partial_\mu\epsilon_\nu) &= \frac{2}{N}(\partial_\nu\partial_\mu + \partial_\mu\partial_\nu)(\partial\cdot\epsilon) \\
2\partial_\mu\partial_\nu(\partial\cdot\epsilon) + \Box\frac{2}{N}(\partial\cdot\epsilon)\,\eta_{\mu\nu} &= \frac{4}{N}\partial_\mu\partial_\nu(\partial\cdot\epsilon) \\
[\Box\eta_{\mu\nu} + (N-2)\,\partial_\mu\partial_\nu](\partial\cdot\epsilon) &= 0. \qquad (3.9)
\end{aligned}$$

Furthermore, by contracting with $\eta^{\mu\nu}$, we find

$$(2N-2)\,\Box(\partial\cdot\epsilon) = 0. \qquad (3.10)$$

From Eq. (3.3)–(3.10), one can derive the explicit form of the conformal transformations in N dimensions.

First, for $N = 1$, the above equation does not impose any constraint on the function $f(x)$, therefore any smooth transformation is conformal in 1D. The case of $N = 2$ will be studied later in the next section. For the moment, we will continue to study the case $N \geq 3$. In this case, Eqs. (3.5) and (3.9) require that the third order derivatives of ϵ vanishe, so that ϵ is at most quadratic in x. Solutions of the deformed Killing equation (3.10) are:

1. Translation $\epsilon^\mu = a^\mu$ generated by $P_\mu = -i\partial_\mu$, where a^μ is a constant vector. The finite form of this transformation is $x^\mu \to x^\mu + a^\mu$.

2. Rotation $\epsilon^\mu = w_{\mu\nu} x^\nu$ generated by $L_{\mu\nu} = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)$, where $w_{\mu\nu}$ is an anti-symmetric second rank tensor. The finite form of this transformation is $x^\mu \to \Lambda^\mu_\nu x^\nu$, where Λ^μ_ν is a Lorentz transformation matrix.

3. Dilatation $\epsilon^\mu = \lambda x^\mu$ generated by $D = -ix \cdot \partial$, where λ is a scaling parameter. The finite form of this transformation is $Kx^\mu \to \lambda x^\mu$, where K is a scaling constant.

4. Special conformal transformation $\epsilon^\mu = b^\mu x^2 - 2x^\mu \left(b \cdot x\right)$ generated by $K_\mu = i\left(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\right)$, where b^μ is a constant vector. The finite form of this transformation is $x^\mu \to \frac{x^\mu - bx^2}{1 - 2(b \cdot x) + b^2 x^2}$.

The commutation relations between each of the group generators represent the algebra of the group:

$$[D, P_\mu] = iP_\mu; \quad [D, K_\mu] = -iK_\mu; \quad [K_\mu, P_\nu] = 2i\left(\eta_{\mu\nu} D - L_{\mu\nu}\right);$$

$$[K_\rho, L_{\mu\nu}] = 2i\left(\eta_{\sigma\mu} K_\nu - \eta_{\sigma\nu} K_\mu\right); \quad [P_\rho, L_{\mu\nu}] = 2i\left(\eta_{\sigma\mu} P_\nu - \eta_{\sigma\nu} P_\mu\right);$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i\left(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}\right). \tag{3.11}$$

The commutation relations above can be put into a simpler form by defining generators J_{ab} which is anti-symmetric in a and b, i.e. $J_{ab} = -J_{ba}$. The indices a and b have two extra numbers compared to μ and ν, i.e. $a, b = (-1, 0, 1, \ldots, N)$. The relation between J_{ab} and its generators (3.11) can be written as the following generators

$$J_{\mu\nu} = L_{\mu\nu}; \quad J_{-1,0} = D; \quad J_{-1,\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right); \quad J_{0,\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right). \tag{3.12}$$

Generators (3.12) obey the commutation relation

$$[J_{ba}, J_{cd}] = i\left(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}\right). \tag{3.13}$$

We see that Eq. (3.13) is similar to the last equation in (3.11). Note that Eq. (3.11) is the Lorentz group in 4D which is isomorphic to $SO\left(3,1\right)$ algebra. Hence the algebra of Eq. (3.13) is $SO\left(N+1,1\right)$ algebra for the metric $\eta_{ab} = \mathrm{diag}\left(-1, 1, 1, \ldots, 1\right)$. This shows the isomorphism between a conformal group in N dimensions and the $SO\left(N+1,1\right)$ group. The number of generators for each translation, rotation, dilatation, and SCT transformation are N, $\frac{1}{2}N(N-1)$, 1, and N respectively. Hence the total number of generators that build a conformal group in N dimensions is

$$\mathcal{N} = \frac{1}{2}\left(N+1\right)\left(N+2\right). \tag{3.14}$$

3.2 Conformal group in $N = 2$ dimensions

In this section, we will discuss the 2D conformal group. We will see that the conformal algebra in 2D has infinitely many generators. In general, we are interested in working in Minkowski spacetime. However, we will work with Euclidean space instead, since it is less burdensome and everything we do in Euclidean space can also be formulated in Minkowski spacetime. From Eq. (3.5) where the indices μ and ν run from 0 to 1, we have:

1. For $\mu = \nu = 0$ and similarly for $\mu = \nu = 1$

$$\begin{aligned}
\partial_0 \epsilon_0 + \partial_0 \epsilon_0 &= \eta_{00} \eta^{\rho\sigma} \partial_\rho \epsilon_\sigma \\
\partial_0 \epsilon_0 + \partial_0 \epsilon_0 &= \eta_{00} \left(\eta^{00} \partial_0 \epsilon_0 + \eta^{11} \partial_1 \epsilon_1 \right) \\
\partial_0 \epsilon_0 + \partial_0 \epsilon_0 &= \partial_0 \epsilon_0 + \partial_1 \epsilon_1 \\
\partial_0 \epsilon_0 &= \partial_1 \epsilon_1 .
\end{aligned} \tag{3.15}$$

2. For $\mu = 0$ and $\mu = 1$

$$\begin{aligned}
\partial_0 \epsilon_1 + \partial_1 \epsilon_0 &= \eta_{01} \left(\eta^{00} \partial_0 \epsilon_0 + \eta^{11} \partial_1 \epsilon_1 \right) \\
\partial_0 \epsilon_1 &= -\partial_1 \epsilon_0 .
\end{aligned} \tag{3.16}$$

We recognize equations (3.15) and (3.16) as the holomorphic and anti-holomorphic Cauchy-Riemann equations in complex analysis. Therefore, we define the Euclidean complex coordinates as

$$z = x^0 + ix^1, \quad \bar{z} = x^0 - ix^1, \tag{3.17}$$

with corresponding holomorphic and anti-holomorphic partial derivatives:

$$\partial_z \equiv \partial = \frac{1}{2}(\partial_0 - i\partial_1), \quad \partial_{\bar{z}} \equiv \bar{\partial} = \frac{1}{2}(\partial_0 + i\partial_1), \tag{3.18}$$

and complex fields

$$\epsilon = \epsilon^0 + i\epsilon^1, \quad \bar{\epsilon} = \epsilon^0 - i\epsilon^1. \tag{3.19}$$

These identifications prompt us to attribute holomorphic and anti-holomorphic functions as left-moving and right-moving functions, respectively. The Euclidean space metric has the form

$$ds^2 = \left(dx^0\right)^2 + \left(dx^1\right)^2 = dz d\bar{z}, \tag{3.20}$$

with $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = 1/2$ are metric tensor components of (3.20).

In Euclidean complex coordinates, conformal transformations of flat space are any holomorphic change of coordinates:

$$z \to z' = f(z), \quad \bar{z} \to \bar{z}' = \bar{f}(\bar{z}), \tag{3.21}$$

where $f(z)$ and $f(\bar{z})$ are arbitrary analytical functions. Under transformations (3.21), the metric (3.20) changes as

$$ds^2 = dz d\bar{z} \to \left| \frac{\partial f(z)}{\partial z} \right|^2 dz d\bar{z} \to \Omega dz d\bar{z}. \tag{3.22}$$

Two-dimensional CFTs are exceptional. Unlike CFTs in higher dimensions, they have an infinite number of conformal transformations. The space of conformal transformations is a finite dimensional group in $N \geq 3$ dimensions. The conformal group of theories defined in $R^{p,q}$ is $SO(p+1, q+1)$ when $p+q \geq 3$.

For simplicity and other intentions, we treat z and $\bar{z}$ as independent coordinates. In this condition, we are projecting the worldsheet from Riemann sphere S^2 to complex plane C^2, which gives us freedom to utilize numerous theorems from complex methods. It should be noted that we are actually working on the real segment $S^2 \subset C^2$ defined by $\bar{z} = z^*$ as shown in Fig. 3.1.

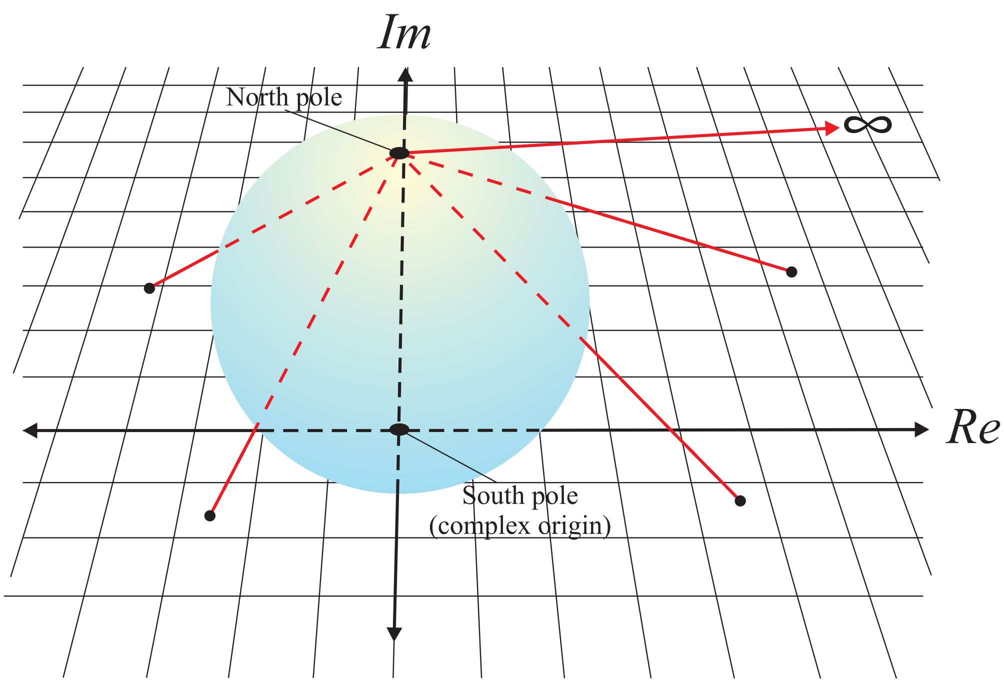


Figure 3.1: Projecting S^2 to C^2 illustration.

The commutation relations of the generators of the conformal algebra can be calculated by taking the infinitesimal transformations

$$z \to z' = z + \epsilon(z), \quad \bar{z} \to \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}), \tag{3.23}$$

where

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} \epsilon_n \left(-z^{n+1}\right), \quad \bar{\epsilon}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{\epsilon}_n \left(-\bar{z}^{n+1}\right), \tag{3.24}$$

are known as the Laurent expansion of $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ at the origin. We consider an infinitesimal change given by $\epsilon_n = -z^{n+1}$ and $\bar{\epsilon}_n = -\bar{z}^{n+1}$ in which the corresponding infinitesimal generators are

$$l_n = -z^{n+1}\partial, \quad \bar{l}_n = -\bar{z}^{n+1}\bar{\partial}. \tag{3.25}$$

Generator (3.25) obeys the following commutation relations:

$$[l_m, l_n] = (m-n)\, l_{m+n}; \quad \left[\bar{l}_m, \bar{l}_n\right] = (m-n)\, \bar{l}_{m+n}; \quad \left[l_m, \bar{l}_n\right] = 0. \tag{3.26}$$

These relations are called the Witt algebra. According to Eq. (3.14), conformal groups in $N = 2$ dimensions have six generators. However, 2D CFTs are exceptional. They have an infinite number of symmetry generators. How are 2D conformal groups built by an infinite number of generators instead of six? Looking into one copy of Witt algebra generators (3.25), for $n < -1$ at $z = 0$, we have singular l_n's. Moreover, by changing coordinate $z = -w^{-1}$, we find

$$l_n = -\left(-\frac{1}{w}\right)^{n+1} (-w)^2 \partial_w = -\left(-\frac{1}{w}\right)^{n+1} \left(-\frac{1}{w}\right)^{-2} \partial_w = -\left(-\frac{1}{w}\right)^{n-1} \partial_w, \tag{3.27}$$

which is singular for $n > 1$ at $w = 0$. Therefore, 2D conformal transformations are defined globally only for the generators $\{l_{-1}, l_0, l_1\}$ as well as $\{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$. Using Eq. (3.25) we find

$$l_{-1} = -\partial_z, \quad l_0 = -z\partial_z, \quad l_1 = -z^2\partial_z, \tag{3.28}$$

which obey the commutation relations

$$[l_0, l_1] = -l_1; \quad [l_0, l_{-1}] = l_{-1}; \quad [l_1, l_{-1}] = 2l_0; \tag{3.29}$$

and similarly for $\{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$. We recognize this as $SL(2, C)$ algebra. The appearance of $SL(2, C)$ group or its subgroup $SL(2, R)$ indicates the existence of 2D conformal group.

3.2.1 The central charge and the Virasoro algebra

The Witt algebra (3.26) can be extended by introducing the central term. This extended version is called the Virasoro algebra which will reduce to Witt one for the vanishing central charge. A central charge is a property of a CFT. It is a measure of the number of degree of freedom of a system. The general form of the Virasoro algebra can be written as

$$[l_n, l_m] = (n - m) l_{n+m} + cp(n, m), \tag{3.30}$$

where c is the central charge and $p(m, n)$ is a function that depends on indices $m, n \in Z$. From the commutation relation $[l_m, l_n] = -[l_n, l_m]$, we see that $p(m, n)$ must be anti-symmetric i.e. $p(m, n) = -p(n, m)$. Furthermore, we redefine

$$\tilde{l}_n \equiv l_n + \frac{cp(n, 0)}{n} \text{ for } n \neq 0, \; \tilde{l}_0 \equiv l_0 + \frac{cp(1, -1)}{2}. \tag{3.31}$$

Next, we will determine the exact form of $p(m, n)$. However, we should first show that $p(n, 0) = 0$ and $p(1, -1) = 0$ by examining the commutation relations. From Eq. (3.30), we know that the commutation relation must be in the form of

$$\left[\tilde{l}_n, \tilde{l}_0\right] = n\tilde{l}_n + cp(n, 0). \tag{3.32}$$

However, we find

$$\begin{aligned}
\left[\tilde{l}_n, \tilde{l}_0\right] &= \tilde{l}_n \tilde{l}_0 - \tilde{l}_0 \tilde{l}_n \\
&= \left(l_n + \frac{cp\,(n,0)}{n}\right)\left(l_0 + \frac{cp\,(1,-1)}{2}\right) - \left(l_0 + \frac{cp\,(1,-1)}{2}\right)\left(l_n + \frac{cp\,(n,0)}{n}\right) \\
&= l_n l_0 - l_0 l_n + l_n \frac{cp\,(1,-1)}{2} + l_0 \frac{cp\,(n,0)}{n} + \frac{c^2 p\,(n,0)\,p\,(1,-1)}{2n} - l_0 \frac{cp\,(n,0)}{n} \\
&\quad - l_n \frac{cp\,(1,-1)}{2} - \frac{c^2 p\,(n,0)\,p\,(1,-1)}{2n} \\
&= [l_n, l_0] = n l_n + cp\,(n,0) = n\tilde{l}_n. \qquad (3.33)
\end{aligned}$$

Matching Eqs. (3.32) and (3.33), it can be seen that $p\,(n,0) = 0$. Similarly, for $n = 1$ and $m = -1$, the commutation relation must be in the form of

$$\left[\tilde{l}_1, \tilde{l}_{-1}\right] = 2\tilde{l}_0 + cp\,(1,-1)\,. \qquad (3.34)$$

However, we find

$$\begin{aligned}
\left[\tilde{l}_1, \tilde{l}_{-1}\right] &= \tilde{l}_1 \tilde{l}_{-1} - \tilde{l}_{-1} \tilde{l}_1 \\
&= \left(l_1 + \frac{cp\,(1,0)}{1}\right)\left(l_{-1} + \frac{cp\,(-1,0)}{-1}\right) - \left(l_{-1} + \frac{cp\,(-1,0)}{-1}\right)\left(l_1 + \frac{cp\,(1,0)}{1}\right) \\
&= l_1 l_{-1} - l_{-1} l_1 + l_1 \frac{cp\,(-1,0)}{-1} + l_{-1} \frac{cp\,(1,0)}{1} + \frac{c^2 p\,(1,0)\,p\,(-1,0)}{-1} - l_{-1} \frac{cp\,(1,0)}{1} \\
&\quad - l_1 \frac{cp\,(-1,0)}{-1} - \frac{c^2 p\,(1,0)\,p\,(-1,0)}{-1} \\
&= [l_1, l_{-1}] = 2l_0 + cp\,(1,-1) = 2\tilde{l}_0. \qquad (3.35)
\end{aligned}$$

Comparing Eqs. (3.34) and (3.35), we can also see that $p\,(1,-1) = 0$. The next step to determine the exact form of $p\,(n,m)$ is to check the following Jacobi identity

$$[[l_m, l_n]\,, l_0] + [[l_0, l_m]\,, l_n] + [[l_n, l_0]\,, l_m] = 0; \qquad (3.36)$$

where

$$[l_m, l_n] = (m-n)\,l_{m+n} + cp\,(m,n)\,; \quad [l_0, l_m] = -m l_m + c\underbrace{p\,(0,m)}_{=0}; \quad [l_n, l_0] = n l_n + c\underbrace{p\,(n,0)}_{=0}. \qquad (3.37)$$

Substituting Eqs. (3.37) to (3.36), we find

$$(m+n)\,p\,(n,m) = 0. \qquad (3.38)$$

From Eq. (3.38), we can see that only if $n \neq -m$ do we get a vanishing $p\,(n,m)$. If $n = -m$, the non-vanishing central extension term is $p\,(n,-n)$, where $n \geq 2$ since $p\,(n,0) = p\,(1,-1) = 0$. By setting $m + n = 1$ and using the Jacobi identity:

$$[[l_{1-n}, l_n]\,, l_{-1}] + [[l_{-1}, l_{1-n}]\,, l_n] + [[l_n, l_{-1}]\,, l_{1-n}] = 0. \qquad (3.39)$$

where

$$[l_{1-n}, l_n] = (1-2n)\, l_1 + cp\,(1-n, n)\,; \tag{3.40}$$

$$[l_{-1}, l_{1-n}] = (n-2)\, l_{-n} + cp\,(-1, 1-n)\,; \tag{3.41}$$

$$[l_n, l_{-1}] = (n+1)\, l_{n-1} + cp\,(n, -1)\,. \tag{3.42}$$

Substituting (3.40)–(3.42) to (3.39) we find

$$p\,(n, -n) - \begin{pmatrix} n+1 \\ n-2 \end{pmatrix} p\,(n-1, 1-n) = 0 \tag{3.43}$$

From Eq. (3.43), it can be seen that for $|n| \geq 2$, $p\,(n, -n)$ will be non-zero. By normalizing $p\,(2, -2) = 1/2$, one finds

$$p\,(n, -n) = \frac{1}{2} \begin{pmatrix} n+1 \\ 3 \end{pmatrix} = \frac{1}{2} \frac{(n+1)\, n\, (n-1)\, \cancel{(n-2)!}}{3!\, \cancel{(n+1-3)!}} = \frac{1}{12} \left(n^3 - n\right). \tag{3.44}$$

Substituting Eq. (3.44) to Eq. (3.30), the Virasoro algebras can be expressed as

$$[l_n, l_m] = (n-m)\, l_{n+m} + \frac{c}{12} \left(n^3 - n\right) \delta_{n+m,0}, \tag{3.45}$$

$$\left[l_n, \bar{l}_m\right] = 0, \tag{3.46}$$

$$\left[\bar{l}_n, \bar{l}_m\right] = (n-m)\, \bar{l}_{n+m} + \frac{c}{12} \left(n^3 - n\right) \delta_{n+m,0}. \tag{3.47}$$

3.2.2 Energy-momentum tensor and Noether currents

In this section, we will review the invariance of some properties of classical theories such as the energy-momentum tensor and Noether currents under conformal transformations (3.1). The energy-momentum tensor, sometimes also called the stress-energy tensor, describes the density and flux of the energy and momentum in spacetime. It is one of the most essential quantities in any field theory. The energy-momentum tensor is defined in the conventional way as the matrix of conserved currents which emerge from the translational invariance

$$\delta x^\mu = \epsilon^\mu, \tag{3.48}$$

where ϵ is a constant parameter. Note that a translation transformation is a particular case of a conformal transformation in Minkowski spacetime.

For the meantime, we continue to work in a flat spacetime, in which, $g_{\mu\nu} = \eta_{\mu\nu}$. Later, we derive conserved currents by promoting the constant ϵ to a function $\epsilon\,(x)$ of the spacetime coordinate. The variation in the action must then be in the form

$$\delta S = \int d^2 x J^\mu \partial_\mu \epsilon, \tag{3.49}$$

where J^μ is an arbitrary function of the fields. This guarantees that $\delta S = 0$ when ϵ is constant, which is the definition of the symmetry. However, the variation of the action must also vanish for all variations of the function $\epsilon\,(x)$, meaning that when the equations of motion are obeyed, the function J^μ must satisfy

$$\partial_\mu J^\mu = 0, \tag{3.50}$$

where now the arbitrary function J^μ can be identified as a conserved current.

Next, we will promote the constant ϵ to a be a function of the spacetime. The variation of the action must be of the form (3.49) and we will see what happens to the conserved current later. Now we consider the theory coupled to gravity in which the background metric $g_{\mu\nu}$ is dynamic. Recall that the dynamical background metric makes the transformations as a diffeomorphism:

$$\delta x^\mu = \epsilon^\alpha (x) . \tag{3.51}$$

The theory is invariant under conformal transformations as long as the variation of the metric satisfies

$$\delta g_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu . \tag{3.52}$$

This implies that if we do the transformation of coordinates in the original theory, then the variation in the action must be the opposite of what we get if we only transform the metric:

$$\delta S = -\int d^2x \frac{\partial S}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = -2 \int d^2x \frac{\partial S}{\partial g_{\mu\nu}} \partial_\mu \epsilon_\nu . \tag{3.53}$$

We have the conserved current from translational invariance and define the energy-momentum tensor as

$$T_{\mu\nu} = -\kappa \frac{\partial S}{\partial g_{\mu\nu}}, \tag{3.54}$$

where κ is the normalization constant that depends on the theory. If we compute the energy-momentum tensor on Minkowski spacetime, then the resulting expression must satisfy the conservation law of energy-momentum $\partial^\mu T_{\mu\nu} = 0$. While on a curved spacetime, the energy-momentum tensor is covariantly conserved $\nabla^\mu T_{\mu\nu} = 0$.

The invariance of the action under conformal transformations yields a very important property of the energy-momentum tensor: tracelessness. Beginning by varying the action with respect to dilatation, which is a particular case of a conformal transformation:

$$\delta g_{\mu\nu} = \epsilon g_{\mu\nu}, \tag{3.55}$$

we then find

$$\delta S = \int d^2x \frac{\partial S}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = -\int d^2x \frac{1}{\kappa} \epsilon T^\mu_\mu . \tag{3.56}$$

However, the change of the action must be zero since dilatation is a symmetry; as a result

$$T^\mu_\mu = 0. \tag{3.57}$$

The tracelessness of the energy-momentum tensor is the main property of a 2D CFT and also higher dimensional CFT. Although many CFTs have this property at the classical level, it is not easy to preserve it at the quantum level.

The energy-momentum tensor transforms under conformal transformation as

$$T_{\mu\nu} \to T'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} T_{\rho\sigma}, \tag{3.58}$$

from the coordinate x^μ to x'^μ. We perform the complex change of coordinates from a Euclidean to a complex one, in which the relation between them are

$$x^0 = \frac{1}{2}(z+\bar{z}), \quad x^1 = -\frac{i}{2}(z-\bar{z}). \tag{3.59}$$

We find that the energy-momentum tensor components in Euclidean coordinates and on the complex plane are

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^0}{\partial z}T_{00} + \frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^1}{\partial z}T_{01} + \frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^0}{\partial \bar{z}}T_{01} + \frac{\partial x^1}{\partial \bar{z}}\frac{\partial x^1}{\partial z}T_1 = \frac{1}{4}(T_{00}+T_{11}) = 0 \tag{3.60}$$

$$T_{zz} = \frac{\partial x^0}{\partial z}\frac{\partial x^0}{\partial z}T_{00} + 2\frac{\partial x^0}{\partial z}\frac{\partial x^1}{\partial z}T_{01} + \frac{\partial x^1}{\partial z}\frac{\partial x^1}{\partial z}T_{11} = \frac{1}{4}(T_{00}+2iT_{01}-T_{11}) = \frac{1}{2}(T_{11}+iT_{01}), \tag{3.61}$$

$$T_{\bar{z}\bar{z}} = \frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^0}{\partial \bar{z}}T_{00} + 2\frac{\partial x^0}{\partial \bar{z}}\frac{\partial x^1}{\partial \bar{z}}T_{01} + \frac{\partial x^1}{\partial \bar{z}}\frac{\partial x^1}{\partial \bar{z}}T_{11} = \frac{1}{4}(T_{00}-2iT_{01}-T_{11}) == \frac{1}{2}(T_{11}-iT_{01}). \tag{3.62}$$

From Eq. (3.60), we know that $T_{00} = -T_{11}$. Using the conservation of the energy-momentum tensor condition we obtain

$$\partial_0 T_{00} + \partial_1 T_{10} = \partial_0 T_{01} + \partial_1 T_{11} = 0. \tag{3.63}$$

Therefore, we are able to show that

$$\partial T_{zz} \neq 0, \quad \bar{\partial} T_{\bar{z}\bar{z}} \neq 0, \tag{3.64}$$

and

$$\bar{\partial} T_{zz} = 0, \quad \partial T_{\bar{z}\bar{z}} = 0. \tag{3.65}$$

This means $T_{zz} = T_{zz}(z)$ is a holomorphic function and $T_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}}(\bar{z})$ is an anti-holomorphic function. It is useful to use the notation:

$$T_{zz}(z) \equiv T(z), \quad T_{\bar{z}\bar{z}}(\bar{z}) \equiv \bar{T}(\bar{z}). \tag{3.66}$$

The energy-momentum tensor presents the Noether currents for translation. First, we consider an infinitesimal change

$$z' = z + \epsilon(z), \quad \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}). \tag{3.67}$$

We will use the same procedure to calculate the current by promoting constant parameter ϵ to a function that depends on the coordinates $\epsilon(z,\bar{z})$. The variation of the action is

$$\begin{aligned}
\delta S &= -\int d^2x \frac{\partial S}{\partial g^{\mu\nu}}\delta g^{\mu\nu} \\
&= \frac{1}{2\pi}\int d^2x T_{\mu\nu}(\partial^\mu \delta x^\nu) \\
&= \frac{1}{2\pi}\int d^2x \frac{1}{2}\left[T_{zz}(\partial^z \delta z) + T_{\bar{z}\bar{z}}(\partial^{\bar{z}}\delta\bar{z})\right] \\
&= \frac{1}{2\pi}\int d^2x (T_{zz}\partial_{\bar{z}}\epsilon + T_{\bar{z}\bar{z}}\partial_z\bar{\epsilon}),
\end{aligned} \tag{3.68}$$

where ϵ and $\bar{\epsilon}$ are holomorphic and anti-holomorphic, respectively, guaranteeing that the variation of the action vanishes.

Next, we consider coordinates z and $\bar{z}$ as independent variables to see separate currents that occur from alterations in both z and $\bar{z}$. By looking at the symmetry

$$\delta z = \epsilon(z), \quad \delta \bar{z} = 0, \tag{3.69}$$

we can interpret the conserved current from Eq. (3.68) by promoting $\epsilon \to \epsilon(z) f(\bar{z})$ for an arbitrary function f, and also the terms which contain $\bar{\partial} f$ as

$$J^z = 0, \quad J^{\bar{z}} = T_{zz}(z)\epsilon(z) \equiv T(z)\epsilon(z). \tag{3.70}$$

Similarly, transformations $\delta \bar{z} = \bar{\epsilon}(\bar{z})$ and $\delta z = 0$, gives us currents

$$\bar{J}^z = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \quad \bar{J}^{\bar{z}} = 0. \tag{3.71}$$

From Eqs. (3.70) and (3.71), we find that currents are also holomorphic and anti-holomorphic, respectively. Note that these currents are conserved. It means that they must satisfy the relation

$$\partial_\mu J^\mu = \partial_z J^z + \partial_{\bar{z}} J^{\bar{z}} = 0. \tag{3.72}$$

3.2.3 Primary fields

A primary field with conformal dimension Δ and planar spin s in 2D spacetime transform as

$$\phi(z, \bar{z}) \to \phi'(w, \bar{w}) = \left|\frac{\partial w}{\partial z}\right|^h \left|\frac{\partial \bar{w}}{\partial \bar{z}}\right|^{\bar{h}} \phi(z, \bar{z}); \tag{3.73}$$

under a conformal transformation $z \to w(z)$ and $\bar{z} \to \bar{w}(\bar{z})$. We specify the right (holomorphic) conformal weight h and left (anti-holomorphic) conformal weight $\bar{h}$, respectively as

$$h = \frac{1}{2}(\Delta + s), \quad \bar{h} = \frac{1}{2}(\Delta - s). \tag{3.74}$$

Both conformal weights h and $\bar{h}$ are real numbers. They describe how operators transform under rotation and scaling. The eigenvalue under rotation is called the spin, $s = h - \bar{h}$, and the eigenvalue under scaling is called the scaling dimension, $\Delta = h + \bar{h}$. If $w(z) \in SL(2, C)/Z_2$, that is, a global conformal group, the field ϕ can be called a quasi-primary field.

Primary fields vary under infinitesimal conformal transformations. If $w(z) = z + \epsilon(z)$ and $\bar{w}(\bar{z}) = \bar{z} + \bar{\epsilon}(\bar{z})$ with $\epsilon \ll 1$ and $\bar{\epsilon} \ll 1$, we can expand

$$\left(\frac{\partial w}{\partial z}\right)^h = \left(\frac{\partial z}{\partial z} + \frac{\partial \epsilon}{\partial z}\right)^h \simeq 1 + h\frac{\partial \epsilon}{\partial z} + \mathcal{O}\left[\left(\frac{\partial \epsilon}{\partial z}\right)^2\right], \tag{3.75}$$

$$\left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} = \left(\frac{\bar{\partial} z}{\bar{\partial} z} + \frac{\partial \bar{\epsilon}}{\partial \bar{z}}\right)^h \simeq 1 + \bar{h}\frac{\partial \bar{\epsilon}}{\partial \bar{z}} + \mathcal{O}\left[\left(\frac{\partial \bar{\epsilon}}{\partial \bar{z}}\right)^2\right], \tag{3.76}$$

$$\phi(z + \epsilon, \bar{z} + \bar{\epsilon}) \simeq \phi(z, \bar{z}) + \epsilon\frac{\partial \phi}{\partial z} + \bar{\epsilon}\frac{\partial \phi}{\partial \bar{z}} + \mathcal{O}\left[\left(\frac{\partial \epsilon}{\partial z}\right)^2\right] + \mathcal{O}\left[\left(\frac{\partial \bar{\epsilon}}{\partial \bar{z}}\right)^2\right], \tag{3.77}$$

and Eq. (3.73) can be rewritten as

$$\phi'(z,\bar{z}) \quad \simeq \quad \left(1+h\frac{\partial\epsilon}{\partial z}\right)\left(1+\bar{h}\frac{\partial\bar{\epsilon}}{\partial\bar{z}}\right)\left(\phi(z,\bar{z})+\epsilon\frac{\partial\phi}{\partial z}+\bar{\epsilon}\frac{\partial\phi}{\partial\bar{z}}\right), \tag{3.78}$$

and the variation of quasi-primary fields is

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) \equiv \phi'(z,\bar{z})-\phi(z,\bar{z}) = \left(h\frac{\partial\epsilon}{\partial z}+\epsilon\frac{\partial}{\partial z}+\bar{h}\frac{\partial\bar{\epsilon}}{\partial\bar{z}}+\bar{\epsilon}\frac{\partial}{\partial\bar{z}}\right)\phi(z,\bar{z}). \tag{3.79}$$

As a matter of fact, a field whose variation under any local 2D conformal transformation is given by Eq. (3.73) or equivalently Eq. (3.79) is called a primary field. Note that all primary fields are quasi-primary ones since they satisfy the global conformal transformation, but not in reverse.

3.2.4 Conformal field theory properties on an infinite cylinder

In this section, we will discuss the cylinder to complex plane mapping. We define the complex coordinate on a cylinder as w and the coordinate on the plane as z. They are related by

$$w = x^0 + ix^1, \quad z = e^{-iw}. \tag{3.80}$$

where x^0 and x^1 are time and space coordinates, respectively. A primary field transforms under mapping (3.80) as

$$\phi_{\text{cyl}}(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h}\left(\frac{\partial\bar{w}}{\partial\bar{z}}\right)^{-\bar{h}}\phi(z,\bar{z}) = z^{-h}\bar{z}^{-\bar{h}}\phi(z,\bar{z}), \tag{3.81}$$

where $\phi_{\text{cyl}}(w,\bar{w})$ is the field defined on the cylinder and $\phi(z,\bar{z})$ is the field defined on the complex plane. Since there is no dependence in $\bar{z}$, the field transformation can be written as

$$\phi_{\text{cyl}}(w) = \left(\frac{\partial z}{\partial w}\right)^{h}\phi(z) = z^h\phi(z), \tag{3.82}$$

or in terms of mode expansion as

$$\phi_{\text{cyl}}(w) = z^h\sum_n \phi_n z^{-n-h} = \sum_n \phi_n e^{-nw}. \tag{3.83}$$

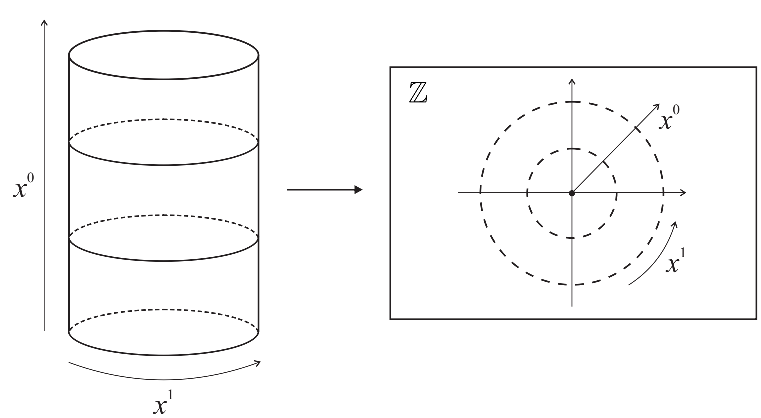


Figure 3.2: Cylinder to complex plane mapping illustration.

An energy-momentum tensor is not a primary field. In the same vein, an energy-momentum tensor on the cylinder transforms under $z \to f(z)$ as

$$T(z) = \left(\frac{\partial f}{\partial z}\right)^2 T(f) + \frac{c}{12} S(f, z), \tag{3.84}$$

where the Schwarzian derivative is defined as

$$S(w, z) = \left(\frac{\partial z}{\partial w}\right)^{-2} \left[\left(\frac{\partial z}{\partial w}\right)\left(\frac{\partial^3 z}{\partial w^3}\right) - \frac{3}{2}\left(\frac{\partial^2 z}{\partial w^2}\right)^2\right]. \tag{3.85}$$

Next, we transform the energy-momentum tensor on the complex plane to the cylinder by setting $z = e^w$, where the Schwarzian derivative becomes $-1/2$. Therefore, Eq. (3.84) becomes

$$T_{\text{cyl}}(w) = \left(\frac{\partial z}{\partial w}\right)^2 T(z) + \frac{c}{12} S(z, w) = z^2 T(z) - \frac{c}{24}. \tag{3.86}$$

On the cylinder, we decompose $T(z)$ in a Laurent expansion

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} l_n, \quad l_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z). \tag{3.87}$$

The Laurent mode expansion of the energy-momentum tensor becomes

$$T(z) = \sum_{n \in \mathbb{Z}} \left(l_n - \frac{c}{24}\delta_{n,0}\right) z^{-n} = \sum_{n \in \mathbb{Z}} l_{n_{\text{cyl}}} z^{-n}. \tag{3.88}$$

The zero mode in the Virasoro generators is now shifted due to the presence of the central charge c:

$$l_{0_{\text{cyl}}} = l_0 - \frac{c}{24}. \tag{3.89}$$

As always, the same statement holds for the anti-holomorphic sector.

3.2.5 Conformal field theory on the torus

In this section, we will discuss CFT defined on the torus and extract constraints on the content of this theory, starting by looking at modular transformations and the partition function.

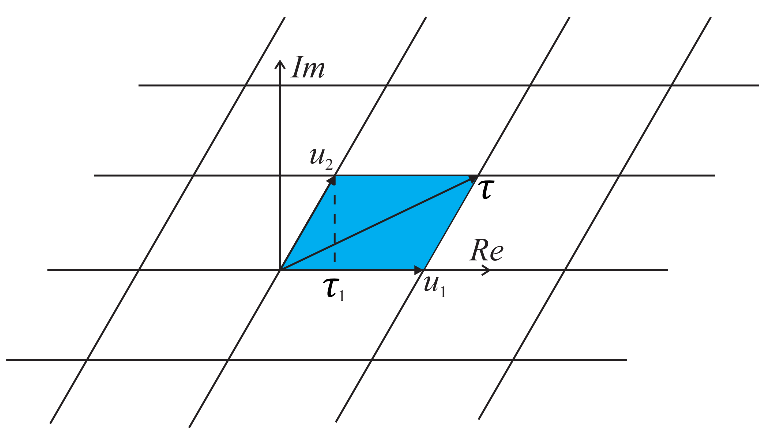


Figure 3.3: Lattice of a torus.

Next, we will discuss CFT properties on a torus and define the partition function in terms of Virasoro generators. We define the coordinate system in which the real and imaginary axes correspond to the spatial and time directions. We also define the the modular parameter of the torus as

$$\tau = \tau_1 + i\tau_2, \tag{3.90}$$

which describe the shape of the torus. In Fig. 3.3, we set $u_1 = 1$ and $u_2 = \tau_1$. In this figure, we see that the time translation with length τ_2 does not come back to the origin. It is shifted in space by the factor of τ_1. In statistical mechanics, the general form of the partition function is

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right), \tag{3.91}$$

where $\beta = 1/k_B T$ and H is the Hamiltonian of the system that generates time translation. Conformal field theory evolves with respect to the complex parameter (3.90). Therefore, we are inspired to define the partition function of CFT on the torus as

$$Z = \mathrm{Tr}\left(e^{-2\pi\tau_2 H} e^{2\pi i \tau_1 P}\right). \tag{3.92}$$

Recalling the T_{00} component of the energy-momentum tensor is the Hamiltonian and the T_{0i} components are the momentum density, we have

$$H = \frac{1}{2\pi}\int dx^1 T_{00} = \frac{1}{2\pi}\oint \left(T_{\mathrm{cyl}}(z)dz + \bar{T}_{\mathrm{cyl}}(\bar{z})d\bar{z}\right), \tag{3.93}$$

$$P = \frac{1}{2\pi}\int dx^1 T_{01} = \frac{i}{2\pi}\oint \left(T_{\mathrm{cyl}}(z)dz - \bar{T}_{\mathrm{cyl}}(\bar{z})d\bar{z}\right), \tag{3.94}$$

respectively. Furthermore, we can write the Hamiltonian and the momentum operator in terms of the Virasoro algebra as

$$H = l_{0_{\mathrm{cyl}}} + \bar{l}_{0_{\mathrm{cyl}}} = l_0 - \frac{c}{24} + \bar{l}_0 - \frac{\bar{c}}{24}, \tag{3.95}$$

$$P = i\left(l_{0_{\mathrm{cyl}}} - \bar{l}_{0_{\mathrm{cyl}}}\right) = i\left(l_0 - \frac{c}{24} - \bar{l}_0 + \frac{\bar{c}}{24}\right). \tag{3.96}$$

Substituting these expressions back into Eq. (3.92), finally the partition function for a CFT on a torus with modular parameter τ can be written as

$$\tilde{Z}(\tau,\bar{\tau}) = \mathrm{Tr}\left(e^{2\pi i\tau\left(l_0 - \frac{c}{24}\right)} e^{-2\pi i\bar{\tau}\left(\bar{l}_0 - \frac{\bar{c}}{24}\right)}\right). \tag{3.97}$$

Setting $q = e^{2\pi i\tau}$ and $\bar{q} = e^{-2\pi i\bar{\tau}}$, Eq. (3.97) becomes

$$\tilde{Z}(\tau,\bar{\tau}) = Z(\tau,\bar{\tau}) q^{-\frac{c}{24}} \bar{q}^{-\frac{\bar{c}}{24}}, \tag{3.98}$$

where $Z(\tau,\bar{\tau}) = \mathrm{Tr}(\varrho(h,\bar{h}))$ and $\varrho(h,\bar{h})$ is the density of state.

3.2.6 Cardy entropy

In general, the modular parameter transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \ ad - bc = 1, \ \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)/Z_2. \tag{3.99}$$

This transformation is called the modular transformation. It has a particular form, which is called the modular S-transformation:

$$S : \tau \to -\frac{1}{\tau} \text{ or } S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.100}$$

The blue-shaded areas in Fig. 3.4 describe the unit lattice of the torus. The invariance of the torus with respect to the S-modular transformation restricts the partition function for a CFT on the torus (3.98) to be invariant under the same transformation, $Z(\tau) = Z(-1/\tau)$. Without the central charge, the partition function on the torus can be written as

$$Z(\tau, \bar{\tau}) = \int dh d\bar{h} e^{2\pi i h \tau} e^{-2\pi i \bar{h} \bar{\tau}} \varrho(h, \bar{h}). \tag{3.101}$$

Using the Fourier transform, Eq. (3.101) becomes

$$\varrho(h, \bar{h}) = \int d\tau d\bar{\tau} e^{-2\pi i h \tau} e^{2\pi i \bar{h} \bar{\tau}} Z(\tau, \bar{\tau}). \tag{3.102}$$

For simplicity, we consider τ and $\bar{\tau}$ as independent variables as well as q and $\bar{q}$. Substituting $q = e^{2\pi i \tau}$ and $\bar{q} = e^{-2\pi i \bar{\tau}}$, the density of state becomes

$$\varrho(h, \bar{h}) = \frac{1}{(2\pi i)^2} \int dq d\bar{q} \frac{\tilde{Z}(q, \bar{q}) q^{\frac{c}{24}} \bar{q}^{\frac{\bar{c}}{24}}}{q^{h+1} \bar{q}^{\bar{h}+1}}. \tag{3.103}$$

We also consider the holomorphic partition function that depends on q for now. We will deal with the anti-holomorphic part later. The partition function is invariant under modular transformation, in which

$$q \to q' = e^{-\frac{2\pi i}{\tau}}, \quad \bar{q} \to \bar{q}' = e^{-\frac{2\pi i}{\bar{\tau}}}, \tag{3.104}$$

so that Eq. (3.103) becomes

$$\varrho(h) = \frac{1}{2\pi i} \int \frac{dq}{q^{h+1}} q^{\frac{c}{24}} \tilde{Z}\left(-\frac{1}{\tau}\right). \tag{3.105}$$

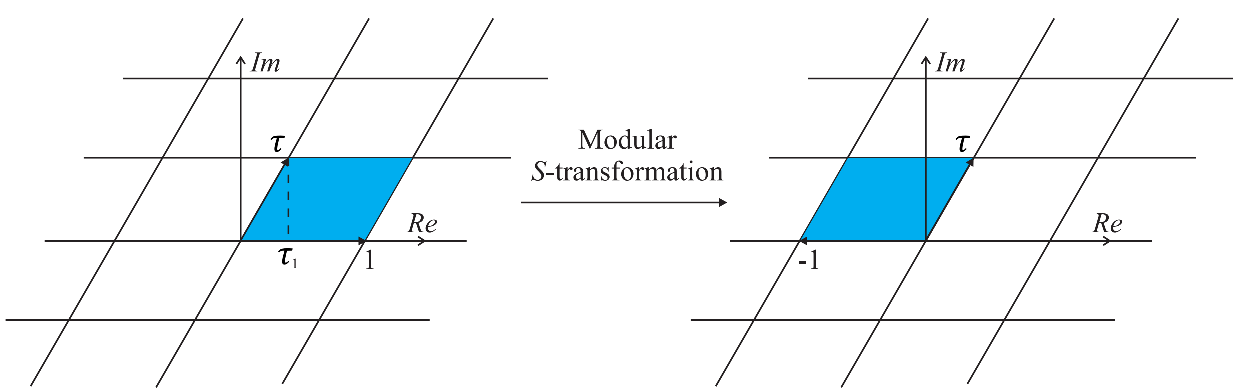


Figure 3.4: Modular S-transformation on lattice of a torus.

We can create the following setup, in which $\tilde{Z}\left(-\frac{1}{\tau}\right) = q'^{-\frac{c}{24}} Z\left(-\frac{1}{\tau}\right)$. Substituting it back to Eq. (3.105), we find the density matrix

$$\begin{aligned}
\varrho(h) &= \frac{1}{2\pi i}\int \frac{dq}{q^{h+1}} q'^{-\frac{c}{24}} Z\left(-\frac{1}{\tau}\right) q^{\frac{c}{24}} \\
&= \frac{1}{2\pi i}\int d\tau e^{-2\pi i\tau(h+1)} e^{-\frac{2\pi i}{\tau}\left(-\frac{c}{24}\right)} e^{2\pi i\tau\left(\frac{c}{24}\right)} 2\pi i e^{2\pi i\tau} Z\left(-\frac{1}{\tau}\right) \\
&= \int d\tau e^{-2\pi i h\tau} e^{\frac{2\pi i c\tau}{24}} e^{\frac{2\pi i c}{24\tau}} Z\left(-\frac{1}{\tau}\right). \qquad (3.106)
\end{aligned}$$

We can apply the saddle point approximation to calculate Eq. (3.106). We define the integral

$$I = \int_{-\infty}^{\infty} dx e^{-f(x)}. \qquad (3.107)$$

The value of $f(x)$ is minimum when $x = x_0$ i.e. $f'(x_0) = 0$. Hence we can apply a Taylor expansion and integral (3.107) becomes

$$\begin{aligned}
I &\simeq \int_{-\infty}^{\infty} dx e^{-f(x_0)-\frac{1}{2}(x-x_0)^2 f''(x_0)} \\
&= e^{-f(x_0)} \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}(x-x_0)^2 f''(x_0)} \\
&= e^{-f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}}. \qquad (3.108)
\end{aligned}$$

We apply this approximation to the density of state (3.106) and find the minimum value of the integral. On the saddle point, in which $\tau \to \infty$, we can approximate $Z\left(-\frac{1}{\tau}\right) \sim e^{-\frac{1}{\tau}}$ to be a constant a. Hence the density of state on the saddle point becomes

$$\varrho(h) \approx a e^{2\pi i} \int_{-\infty}^{\infty} d\tau e^{-h\tau + \frac{c\tau}{24} + \frac{c}{24\tau}}, \qquad (3.109)$$

where $f(\tau) = -h\tau + \frac{c}{24\tau} + \frac{c\tau}{24}$. From the condition, $\frac{df(\tau)}{d\tau} = 0$, and assuming $h \gg c$, we find

$$\tau \approx i\sqrt{\frac{c}{24h}}. \qquad (3.110)$$

Substituting Eq. (3.110) to the density matrix (3.106), we obtain

$$\begin{aligned}
\varrho(h) &\approx e^{2\pi i^2 h\sqrt{\frac{c}{24h}}} e^{\frac{2\pi i c}{24 i\sqrt{\frac{c}{24h}}}} e^{\frac{2\pi i^2 c}{24}\sqrt{\frac{c}{24h}}} \\
&= e^{2\pi\sqrt{\frac{ch}{24}}} e^{\frac{2\pi c}{24}\sqrt{\frac{24h}{c}}} e^{-\frac{2\pi c}{24}\sqrt{\frac{c}{24h}}} \\
&= e^{2\pi\sqrt{\frac{ch}{24}}} e^{2\pi\sqrt{\frac{ch}{24}}} \\
&= e^{2\pi\sqrt{\frac{ch}{6}}}. \qquad (3.111)
\end{aligned}$$

Note that we have the eigen-equations

$$l_0 |h, \bar{h}\rangle = h |h, \bar{h}\rangle, \ \bar{l}_0 |h, \bar{h}\rangle = \bar{h} |h, \bar{h}\rangle. \qquad (3.112)$$

Considering the anti-holomorphic part in the theory, the Cardy entropy formula can be written as

$$S_{\mathrm{CFT}} = \ln \varrho(h, \bar{h}) = 2\pi\left(\sqrt{\frac{c l_0}{6}} + \sqrt{\frac{\bar{c}\bar{l}_0}{6}}\right). \qquad (3.113)$$

We use terminologies in which the holomorphic part is called the left-mover of the theory with the corresponding central charge is c_L and the anti-holomorphic part is the right-mover of the theory with the corresponding central charge is c_R. From the eigen-equations (3.112), the eigenvalue of l_0 is energy E. Therefore the Cardy formula can also be written as

$$S_{\text{CFT}} = 2\pi\left(\sqrt{\frac{c_L E_L}{6}} + \sqrt{\frac{c_R E_R}{6}}\right). \tag{3.114}$$

Using the first law of thermodynamics relation $dE = TdS$, again we can rewrite the Cardy entropy formula as

$$S_{\text{CFT}} = \frac{\pi^2}{3}\left(c_L T_L + c_R T_R\right). \tag{3.115}$$

This form of Cardy entropy formula of a 2D CFT is mainly used in the discussion of Kerr/CFT correspondence to find the microscopic entropy of the black hole.

Moreover, in the last equation, we see the left (L) and right (R) expressions. These expressions are comparable to the left and right moving of travelling waves $y_1 = A\sin(kx + \omega t)$ and $y_2 = A\sin(kx - \omega t)$, respectively. Setting $x^0 = kx$, $x^1 = -i\omega t$, we can write the wave equations as a holomorphic and anti-holomorphic functions $y_1 = A\sin(z)$ and $y_2 = A\sin(\bar{z})$, respectively.

4 Holographic Aspects of Kerr Black Holes

The holographic description of a black hole is an interesting topic in gravitational physics. There are two well-established ways to get the holographic description. One is to study the asymptotic symmetry group of the near-horizon geometry of an extremal black hole to read the central charges. The other way is to find the hidden conformal symmetry in the scalar scattering off a non-extremal black hole. In this chapter, we review the Kerr/CFT correspondence for both extremal and non-extremal Kerr black holes. The main references of this chapter are Refs. [19, 20, 41, 46, 71, 72, 73, 74, 75, 76].

4.1 Extremal Kerr black hole and holography

4.1.1 AdS$_3$ and $SL(2,R)_L \times SL(2,R)_R$

An anti-de Sitter space is a spacetime with a constant negative curvature. It is the maximally symmetric solution of the Einstein's field equation with a negative cosmological constant. AdS$_D$ can be realized as a hyperboloid embedded in a $(D+1)$-dimensional geometry. In this section we will discuss AdS$_3$ which is obtained as the hyperboloid

$$-U^2 - V^2 + X^2 + Y^2 = -L^2, \tag{4.1}$$

with radius L embedded in maximally symmetric spacetime in flat $R^{2,2}$ in coordinates (U, V, X, Y) with line element

$$ds^2 = -dU^2 - dV^2 + dX^2 + dY^2. \tag{4.2}$$

Comparing the line element (4.2) to the Euclidean space

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{4.3}$$

where the isometries form the rotational group $SO(3)$, and with Minkowski spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{4.4}$$

where the isometries are the Lorentz transformation which form the group $SO(1,3)$, we can clearly see that the isometries of AdS$_3$ form the group $SO(2,2)$. This is one of the special orthogonal groups that can be represented by matrices $\mathcal{M}$ with determinant one and satisfy the orthogonality relation $\mathcal{M}\mathcal{M}^T = 1$. We point out that isometries are mapping that preserve distance. The isometry group is the group of all isometries of the space onto itself. As a Lie algebra, note that

$$SO(2,2) = SL(2,R)_L \times SL(2,R)_R. \tag{4.5}$$

This is a special feature of AdS_3. In general, the AdS_D isometry group is $SO(D-1,2)$ which does not split into two factors.

4.1.2 $SL(2,R)$ Lie algebra

Previously, we have seen that $SO(2,2)$ is a Lie group. It is a group that depends on continuous parameters. A Lie group is a smooth manifold, in our case, the hyperboloid in $R^{2,2}$. A Lie group gives rise to A Lie algebra, which is a vector space together with an operation called the Lie bracket. For instance, in quantum mechanics, the angular momentum operators L_x, L_y, L_z with their commutators correspond to the Lie group $SO(3)$. These operators are the generators of the algebra. The generators of $SO(2,2)$ are

$$J_{01} = V\partial_U - U\partial_V, \quad J_{02} = X\partial_V - V\partial_X, \quad J_{03} = Y\partial_V - V\partial_Y,$$
$$J_{23} = X\partial_Y - Y\partial_X, \quad J_{12} = X\partial_U - U\partial_X, \quad J_{13} = Y\partial_U - U\partial_Y. \tag{4.6}$$

We can construct two commuting $SL(2,R)$ factors from these generators. From $SL(2,R)_L$ we have

$$L_1 = \frac{1}{2}(J_{01} + J_{23}), \quad L_2 = \frac{1}{2}(J_{02} - J_{13}), \quad L_3 = \frac{1}{2}(J_{12} + J_{03}), \tag{4.7}$$

and from $SL(2,R)_R$ we have

$$\bar{L}_1 = \frac{1}{2}(J_{01} - J_{23}), \quad \bar{L}_2 = \frac{1}{2}(J_{02} + J_{13}), \quad \bar{L}_3 = \frac{1}{2}(J_{12} - J_{03}), \tag{4.8}$$

With these definitions, it is straightforward to show that the operators $\{L_1, L_2, L_3\}$ obey the commutation relation

$$[L_1, L_2] = -L_3; \quad [L_1, L_3] = L_2; \quad [L_2, L_3] = L_1; \tag{4.9}$$

and similarly for $\{\bar{L}_1, \bar{L}_2, \bar{L}_3\}$. From $\{L_1, L_2, L_3\}$, we can construct the linear combinations $\{L_0, L_\pm\}$ that obey the algebra

$$[L_+, L_-] = 2L_0; \quad [L_0, L_\pm] = \mp L_\pm. \tag{4.10}$$

and similarly for $\{\bar{L}_1, \bar{L}_2, \bar{L}_3\}$. This is exactly the Lie algebra for $SL(2,R)$. A Casimir is an important operator in Lie algebra. It is an operator that commutes with all other operators in the algebra. In $SO(3)$ group, the Casimir is $L^2 = {L_x}^2 + {L_y}^2 + {L_z}^2$. The Casimir of $SL(2,R)$ is then

$$L^2 = \frac{1}{2}\left(L_+L_- + L_-L_+\right) - {L_0}^2, \tag{4.11}$$

and we have

$$L^2\Phi = h(h-1)\Phi, \tag{4.12}$$

where h is the weight of the field Φ. Since there are two commuting copies of $SL(2,R)$, the Casimir L^2 is the same for both copies. Thus the weight h_L and h_R are the same.

In Chapter 3.2, we discuss the 2D CFT. It is important to note that from Eq. (3.29), we recognize exactly the $SL(2,R)$ algebra. This means that the global conformal group of 2D CFT has the same $SL(2,R)_L \times SL(2,R)_R$ symmetries as AdS_3!

4.1.3 $\mathbf{AdS_3/CFT_2}$ correspondence

In the previous section, we found that AdS_3 and CFT_2 have the same $SL(2,R)_L \times SL(2,R)_R$ symmetries. This is one of the properties of $\text{AdS}_3/\text{CFT}_2$ correspondence. First, we define Poincarè coordinates as

$$U = \frac{1}{2r}(L^2 + x^2 + r^2 - t^2), \quad V = L\frac{t}{r}, \quad Y = \frac{1}{2r}(-L^2 + x^2 + r^2 - t^2), \quad X = L\frac{x}{r}. \tag{4.13}$$

Substituting these coordinates to the metric (4.2) we have

$$ds^2 = \frac{L^2}{r^2}(-dt^2 + dx^2 + dr^2). \tag{4.14}$$

where the range of coordinates is given by $-\infty < t < \infty$ and $-\infty < x < \infty$. However, the radial coordinate r splits the hyperboloid into two patches:

$$U - Y = \frac{L^2}{r}, \tag{4.15}$$

where the first Poincaré patch is at $r > 0$, ergo $U > -Y$ and the second patch is at $r > 0$, ergo $U < -Y$. From (4.10), a basis $\{L_0, L_\pm\}$ for $SL(2,R)_L$ is

$$L_0 = -L_2, \quad L_\pm = i(L_1 \pm L_3). \tag{4.16}$$

Substituting coordinates (4.13) into the six linearly independent Killing vectors (4.6) and using generators (4.7), we find

$$L_0 = -\frac{r}{2}\partial_r - z\partial_z, \quad L_- = iL\partial_z, \quad L_+ = -\frac{i}{L}(zr\partial_r + z^2\partial_z + r^2\partial_{\bar{z}}), \tag{4.17}$$

where $z = t + x$ and $\bar{z} = t - x$. The CFT lives at the boundary in which $r = 0$. Thus the generators above become

$$L_0 = -z\partial_z, \quad L_- = iL\partial_z, \quad L_+ = -\frac{iz^2}{L}\partial_z, \tag{4.18}$$

where these are the same as generators of conformal transformation (3.28). This is not surprising since we know that AdS_3 is dual to CFT_2. Note that we see the factors i and L since they are in Minkowski flat space and z is a dimensional coordinate.

4.1.4 Kerr/CFT correspondence

In the extremal limit of $a = M$, the horizon radius, the angular velocity, the Hawking temperature, and the Bekenstein-Hawking entropy of Kerr black hole (2.41), respectively, simplify to

$$r_\pm = a = M, \quad S = 2\pi M^2 = 2\pi J, \quad T_H = 0, \quad \Omega_H = \frac{a}{2Mr_+} = \frac{1}{2M}, \tag{4.19}$$

and the line element reduces to

$$ds^2 = -\frac{\Delta}{\Sigma}\left(d\hat{t} - a\sin^2\theta d\hat{\phi}\right)^2 + \frac{\Sigma}{\Delta}d\hat{r}^2 + \frac{\sin^2\theta}{\Sigma}\left[(\hat{r}^2 + a^2)d\hat{\phi} - ad\hat{t}\right]^2 + \Sigma d\theta^2, \tag{4.20}$$

where

$$\Delta = (\hat{r} - a)^2, \quad \Sigma = \hat{r}^2 + a^2\cos^2\theta. \tag{4.21}$$

We want to study the region near the extremal Kerr horizon at $\hat{r} = M$. Following Bardeen and Horowitz and zooming in on the near-horizon region $\hat{r} = M$, we may take the near-horizon limit by introducing the new dimensionless coordinates:

$$r = \frac{\hat{r} - M}{\lambda M}, \quad t = \frac{\lambda \hat{t}}{2M}, \quad \phi = \hat{\phi} - \frac{\hat{t}}{2M}, \tag{4.22}$$

where we set $\lambda \to 0$ in order to keep coordinates (t, y, ϕ, θ). The result is the near-horizon extremal Kerr or NHEK geometry:

$$ds^2 = \Gamma(\theta) \left[\frac{dr^2}{r^2} + \alpha(\theta)^2 d\theta^2 - r^2 dt^2 + \Lambda(\theta)^2 \left(d\phi + k r dt \right)^2 \right], \tag{4.23}$$

where

$$\alpha(\theta) = 1, \quad \Gamma(\theta) = J(1 + \cos^2\theta), \quad \Lambda(\theta) = \frac{2\sin\theta}{1 + \cos^2\theta}, \quad k = 1, \tag{4.24}$$

$\phi \sim \phi + 2\pi$ and $0 \leq \theta \leq \pi$. The NHEK metric is not asymptotically flat, thus the metric does not reduce to the Minkowski metric at $r \to \infty$. This NHEK geometry has an enhanced isometry group $SL(2,R)_R \times U(1)_L$ (∂_t is in $SL(2,R)_R$ and ∂_ϕ is the $U(1)_L$). The rotational $U(1)_L$ isometry group is generated by the Killing vector

$$\xi_0 = -\partial_\phi, \tag{4.25}$$

and the time translations become part of an enhanced $SL(2,R)_R$ isometry group generated by the Killing vectors:

$$\tilde{L}_{-1} = \partial_t, \quad \tilde{L}_0 = t\partial_t - r\partial_r, \quad \tilde{L}_1 = \frac{1}{2r^2}\partial_t - \frac{t^2}{2}\partial_r - tr\partial_r - \frac{1}{r}\partial_\phi, \tag{4.26}$$

which obey the $SL(2,R)$ algebra. Fixing polar angle $\theta = \theta_0$ and setting the special value $\Lambda(\theta) = 1$ one can recover a 3D geometry which is a *warped* version of AdS$_3$:

$$ds^2 = \Gamma(\theta) \left[\frac{dr^2}{r^2} - r^2 dt^2 + \left(d\phi + r dt \right)^2 \right]. \tag{4.27}$$

Since we know that gravity on AdS$_3$ has a conformal symmetry, this is a strong clue that extremal Kerr black holes are dual to 2D CFT. The Kerr/CFT correspondence can be established by matching the microscopic CFT Cardy entropy (3.115) to the macroscopic Bekenstein-Hawking entropy of the Kerr black holes (2.89). Prior to that, we need the formula of the central charge. It can be derived from the asymptotic symmetry group analysis of the near-horizon extremal black hole geometry. However, in this thesis, we focus on finding the holographic description of a non-extremal black hole. Thus, we do not provide the detailed analysis of the asymptotic symmetry group. The detailed derivation of the asymptotic symmetry group can be found, for example in Ref. [75]. The result of the asymptotic symmetry group analysis shows that there is an exact $SL(2,R)_L \times SL(2,R)_R$ symmetry, and an additional asymptotic Virasoro symmetry

$$[l_n, l_m] = (n - m)\, l_{n+m} + \frac{J}{\hbar} \left(n^3 - n \right) \delta_{n+m,0}. \tag{4.28}$$

Comparing this result to the Virasoro algebra (3.45), we can read the central charge for the NHEK geometry as

$$c = \frac{12J}{\hbar}, \tag{4.29}$$

where we have restored the factor of $\hbar$ normally set to one. It turns out that the central charge can also be expressed in the following integral:

$$c = 3k \int_0^{\pi} d\theta \sqrt{\alpha(\theta)\Gamma(\theta)\Lambda(\theta)} = 12J. \tag{4.30}$$

The temperature of the CFT is the thermodynamic potential dual to the zero mode of the Virasoro algebra. From the first law of black hole thermodynamics, the CFT must be at the left-moving temperature:

$$T_L = \frac{1}{2\pi}. \tag{4.31}$$

Finally, using the Cardy entropy formula (3.115) for the 2D CFT, we find the entropy obeys

$$S_{\rm CFT} = \frac{\pi^2}{3} c T_L = 2\pi J, \tag{4.32}$$

which match exactly the Bekenstein-Hawking entropy (2.89)!

Away from extremality, however, we cannot indicate any conformal symmetries at the near-horizon region of the Kerr black holes. Rindler space, which is the spacetime seen by an observer with constant proper acceleration, is the genuine near-horizon geometry of the non-extremal spacetime and it does not have any associated conformal symmetries. In other words, the conformal symmetries are not the symmetries of the non-extremal Kerr black hole geometry (as they are for the case of the extremal Kerr black holes).

4.2 Non-extremal Kerr black hole and holography

4.2.1 Scalar scattering off a Kerr black hole

It is surprising that away from extremality, the $SL(2,R)_L \times SL(2,R)_R$ invariance is present in the dynamics of scalar field perturbations around the Kerr black hole in a specific regime, at the low energy limit and near the black hole horizon. In this regime, the scalar field perturbation equation can be written as a $SL(2,R)_L \times SL(2,R)_R$ Casimir eigen-equation. In other words, the local hidden conformal symmetries are non-geometric but appear in the probe dynamics. We start by considering a massless scalar particle field Ψ outside of a Kerr black hole that obeys the following KG wave equation:

$$\frac{1}{\sqrt{-g}} \partial_\alpha \left(g^{\alpha\beta} \sqrt{-g} \partial_\beta \Psi \right) = 0. \tag{4.33}$$

Since the Kerr metric has axial and time translational symmetry (the Killing vectors are ∂_t and ∂_ϕ), we can expand the field in eigen-modes

$$\Psi\left(t, r, \theta, \phi\right) = e^{i(m\phi - \omega t)} \Psi\left(r, \theta\right), \tag{4.34}$$

where ω is the frequency of the field and m is the angular momentum projection quantum number. Initially, it was unexpected that scalar wave equation in Kerr spacetime would be separable. Writing $\Psi\left(r,\theta\right) = R\left(r\right) S\left(\theta\right)$, Eq. (4.33) can famously [77] be separated. This gives us two separated equations, specifically the angular part

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dS\left(\theta\right)}{d\theta} \right) + \left(\lambda + a^2 \omega^2 \cos^2\theta - \frac{m^2}{\sin^2\theta} \right) S\left(\theta\right) = 0, \tag{4.35}$$

and the radial part

$$\frac{d}{dr}\left(\Delta\frac{dR\left(r\right)}{dr}\right)+\left(\frac{G^2}{\Delta}+2am\omega-\lambda\right)R\left(r\right)=0, \tag{4.36}$$

with $G=\omega\left(r^2+a^2\right)-am$ and the separation constant λ. In order to clearly show the divergences at the outer and inner horizons, the radial equation (4.36) can be rearranged to

$$\left[\partial_r\Delta\partial_r+\frac{(2Mr_+\omega-am)^2}{(r-r_+)(r_+-r_-)}-\frac{2Mr_-\omega-am)^2}{(r-r_-)(r_+-r_-)}+g\left(r\right)\right]R(r)=\lambda R(r), \tag{4.37}$$

where $g\left(r\right)=\left[\Delta+4M(r+M)\right]\omega^2$. To simplify the equation of motion (4.37). We consider the low frequency scalar field limit $\omega\ll 1/M$ and the near region geometry defined by $r\ll 1/\omega$. With these assumptions, one can neglect $g\left(r\right)$.

4.2.2 Hidden conformal symmetry

In order to study the hidden conformal symmetry, first we introduce the following conformal coordinates for non-extremal Kerr black holes:

$$w^+=\sqrt{\frac{r-r_+}{r-r_-}}e^{2\pi T_R\phi},\quad w^-=\sqrt{\frac{r-r_+}{r-r_-}}e^{2\pi T_L\phi-\frac{t}{2M}},\quad y=\sqrt{\frac{r_+-r_-}{r-r_-}}e^{\pi(T_L+T_R)\phi-\frac{t}{4M}}, \tag{4.38}$$

where the right and left temperatures of the CFT are, respectively,

$$T_R\equiv\frac{r_+-r_-}{4\pi a},\quad T_L\equiv\frac{r_++r_-}{4\pi a}. \tag{4.39}$$

Now we define locally the vector fields

$$H_1=i\partial_+,\quad H_0=i(w^+\partial_++\frac{1}{2}y\partial_y),\quad H_{-1}=i(w^{+2}\partial_++w^+y\partial_y-y^2\partial_-), \tag{4.40}$$

and

$$\bar{H}_1=i\partial_-,\quad \bar{H}_0=i(w^-\partial_-+\frac{1}{2}y\partial_y),\quad \bar{H}_{-1}=i(w^{-2}\partial_-+w^-y\partial_y-y^2\partial_+). \tag{4.41}$$

These vector fields obey the algebra of $SL(2,R)$ group:

$$[H_0,H_{\pm1}]=\mp iH_{\pm1};\quad [H_{-1},H_1]=-2iH_0;\quad [\bar{H}_0,\bar{H}_{\pm1}]=\mp i\bar{H}_{\pm1};\quad [\bar{H}_{-1},\bar{H}_1]=-2i\bar{H}_0. \tag{4.42}$$

Because vector fields (4.40)–(4.41) obey the $SL(2,R)$ algebra and related to conformal symmetries, we call coordinates (4.38) *conformal*. The $SL(2,R)$ quadratic Casimir is

$$\mathcal{H}^2=\bar{\mathcal{H}}^2=-H_0^2+\frac{1}{2}(H_1H_{-1}+H_{-1}H_1)=\frac{1}{4}(y^2\partial_y^2-y\partial_y)+y^2\partial_+\partial_-. \tag{4.43}$$

However, we want to compute the Casimir in terms of the coordinates (t,r,ϕ) to reproduce the radial equation in the near region. Therefore, we need to write Eqs. (4.40)–(4.41) and (4.43) in terms of the (t,r,ϕ) coordinates. Note that we can eliminate ϕ and t to get $r\left(w^+,w^-,y\right)$:

$$\frac{w^+w^-}{y^2}=\frac{r-r_+}{r_+-r_-}, \tag{4.44}$$

so that we can find

$$r = \frac{w^+ w^-}{y^2}(r_+ - r_-) + r_+. \tag{4.45}$$

From the expression of w^+, one finds

$$\phi = \frac{1}{4\pi T_R}\log\left(w^+ \frac{w^+ w^- + y^2}{w^-}\right). \tag{4.46}$$

Finally we get $t(w^+, w^-, y)$ from $w^+/w^- = e^{2\pi(T_R - T_L)\phi + \frac{t}{2M}}$:

$$\begin{aligned} t &= 2M\left[\log\left(\frac{w^+}{w^-}\right) - 2\pi(T_R - T_L)\phi\right] \\ &= M\log\left[\frac{w^+}{w^-(w^+ w^- + y^2)}\right] + M\frac{T_L}{T_R}\log\left[\frac{w^+(w^+ w^- + y^2)}{w^-}\right]. \end{aligned} \tag{4.47}$$

Taking the partial derivatives and using the chain rule, we can construct

$$\partial_+ = e^{-2\pi T_R\phi}\left(\Delta^{1/2}\partial_r + \frac{1}{2\pi T_R}r - \frac{M}{\Delta^{1/2}}\partial_\phi + \frac{2T_L}{T_R}\frac{Mr - a^2}{\Delta^{1/2}}\partial_t\right), \tag{4.48}$$

$$\partial_- = e^{-2\pi T_L\phi + \frac{t}{2M}}\left(\Delta^{1/2}\partial_r - \frac{a}{\Delta^{1/2}}\partial_\phi - 2M\frac{r}{\Delta^{1/2}}\partial_t\right), \tag{4.49}$$

$$y\partial_y = -2(r - r_+)\partial_r + \frac{1}{2\pi T_R}\frac{r_+ - r_-}{r - r_-}\partial_\phi + 2M\frac{2r_-}{r - r_-}\partial_t. \tag{4.50}$$

Substituting Eqs. (4.45)–(4.50) into (4.40)–(4.41) one finds

$$H_1 = ie^{-2\pi T_R\phi}\left(\Delta^{1/2}\partial_r + \frac{1}{2\pi T_R}\frac{r - M}{\Delta^{1/2}}\partial_\phi + \frac{2T_L}{T_R}\frac{Mr - a^2}{\Delta^{1/2}}\partial_t\right), \tag{4.51}$$

$$H_0 = \frac{i}{2\pi T_R}\partial_\phi + 2iM\frac{T_L}{T_R}\partial_t, \tag{4.52}$$

$$H_{-1} = ie^{2\pi T_R\phi}\left(-\Delta^{1/2}\partial_r + \frac{1}{2\pi T_R}\frac{r - M}{\Delta^{1/2}}\partial_\phi + \frac{2T_L}{T_R}\frac{Mr - a^2}{\Delta^{1/2}}\partial_t\right), \tag{4.53}$$

and

$$\bar{H}_1 = ie^{-2\pi T_L\phi + \frac{t}{2M}}\left(\Delta^{1/2}\partial_r - \frac{a}{\Delta^{1/2}}\partial_\phi - 2M\frac{r}{\Delta^{1/2}}\partial_t\right), \tag{4.54}$$

$$\bar{H}_0 = -2iM\partial_t, \tag{4.55}$$

$$\bar{H}_{-1} = ie^{2\pi T_L\phi - \frac{t}{2M}}\left(-\Delta^{1/2}\partial_r - \frac{a}{\Delta^{1/2}}\partial_\phi - 2M\frac{r}{\Delta^{1/2}}\partial_t\right), \tag{4.56}$$

and the Casimir in terms of (t, r, ϕ) becomes

$$\mathcal{H}^2 = \bar{\mathcal{H}}^2 = \partial_r\Delta\partial_r - \frac{(2Mr_+\partial_t + a\partial_\phi)^2}{(r - r_+)(r_+ - r_-)} + \frac{(2Mr_-\partial_t + a\partial_\phi)^2}{(r - r_-)(r_+ - r_-)}. \tag{4.57}$$

The near region radial wave equation (4.37) can be written as

$$\bar{\mathcal{H}}^2\Psi = \mathcal{H}^2\Psi = l(l+1)\Psi. \tag{4.58}$$

We see that the scalar Laplacian has reduced to the $SL(2,R)$ Casimir. We also see that the weight of the field Ψ is the same for both $SL(2,R)_L$ and $SL(2,R)_R$:

$$(h_L, h_R) = (l, l). \tag{4.59}$$

From this result, we might think that we can describe the near region by a dual 2D CFT. It is not accurate, however, that the solutions of Kerr wave equation in the near region form an $SL(2,R)$ representation because vector fields (4.51)–(4.56) are locally defined, not globally. We will interpret the result in the next section.

4.2.3 Temperature and entropy in conformal field theory

We mentioned before that the vector fields are not globally defined. The vector fields are not periodic under the angular identification

$$\phi \sim \phi + 2\pi. \tag{4.60}$$

These symmetries cannot be used to generate a new global solution from the old ones. This can be interpreted as the statement that the $SL(2,R)_L \times SL(2,R)_R$ symmetry is spontaneously broken under the angular identification (4.60), under which the conformal coordinates are identified as

$$w^+ \sim e^{4\pi^2 T_R} w^+, \quad w^- \sim e^{4\pi^2 T_L} w^-, \quad y \sim e^{2\pi^2 (T_L + T_R)} y. \tag{4.61}$$

These identifications are generated by the element of $SL(2,R)_L \times SL(2,R)_R$ group:

$$g = e^{-4\pi^2 i T_R H_0 - 4\pi^2 i T_L \bar{H}_0}, \tag{4.62}$$

which is exactly the form of the identification for a CFT partition function at finite temperatures T_L and T_R. At a fixed radius, from (4.38), we can write the relation between conformal coordinates (w^+, w^-) and Boyer-Lindquist coordinates (t, ϕ) as

$$w^\pm = e^{\pm t^\pm}, \tag{4.63}$$

where

$$t^+ = 2\pi T_R \phi, \quad t^- = \frac{t}{2M} - 2\pi T_L \phi. \tag{4.64}$$

This is the relation between Minkowski ($w^\pm$) and Rindler ($t^\pm$) coordinates. Under the periodic identification (4.60), the Rindler coordinates have the identification

$$t^+ \sim t^+ + 4\pi^2 T_R, \quad t^- \sim t^- - 4\pi^2 T_L. \tag{4.65}$$

Observing from Minkowski vacuum by tracing over the quantum state, we will get a thermal density matrix at temperatures T_L and T_R. Therefore, Kerr black holes should be dual to a mixed thermal state with finite temperatures T_L and T_R in the dual 2D CFT.

As a universal check of this duality, we would like to microscopically reproduce the Bekenstein-Hawking entropy of Kerr black holes using the Cardy formula for the dual 2D CFT. The central charges for near-extremal Kerr black hole are

$$c_R = c_L = 12J, \tag{4.66}$$

where we assume that the conformal symmetry for the extremal black hole case connects smoothly to that of the near-extremal black hole. Thus, the central charge is still given by Eq. (4.29). We have two copies of the same $SL\,(2,R)$ algebra, with left and right temperatures T_L and T_R. Thus by using the Cardy formula

$$S_{\text{CFT}} = \frac{\pi^2}{3} \left(c_L T_L + c_R T_R \right), \tag{4.67}$$

together with the CFT left and right temperatures (4.39), the CFT entropy is

$$S_{\mathrm{CFT}} = \frac{2\pi r_+ J}{a} = 2\pi M r_+ = S_{\mathrm{BH}}. \tag{4.68}$$

Comparing this to the Bekenstein-Hawking entropy (2.89), we see a detailed match for the non-extremal Kerr black holes. The Cardy formula is a measure for the microscopic degeneracy associated to a thermal state. Therefore we can interpret the black hole entropy as a measure of the number of microstates in the dual CFT.

5 Holographic Aspects of Black Holes in Modified Gravity

In this chapter, we study the holographic description of black hole in a type of scalar-tensor gravity and $f(T)$ gravity by finding the hidden conformal symmetry in the scalar scattering off a black hole. The material in chapter 5.2 is based on our published work [78].

5.1 Is there a hidden conformal symmetry for black holes in scalar-tensor gravity?

5.1.1 Scalar scattering off a Janis-Newman-Winicour black hole

Rotating JNW spacetime describes a rotating mass in Einstein's theory of gravity minimally coupled to a real scalar field. The Schwarzschild spacetime is a vacuum solution to the Einstein field equations after we impose that spacetime is spherically symmetric and asymptotically flat. If we introduce a real scalar field rather than vacuum spacetime, we obtain the non-rotating JNW solution [79]. The rotating JNW solution was originally derived in Ref. [80]. In the limit of vanishing scalar charge, the Kerr metric is obtained. This makes the rotating JNW solution one of the generalizations of the Kerr metric. Intriguingly, the rotating JNW metric does not posses any event horizon and there is a surface-like singularity where the scalar curvature diverges.

The influential theorems of Penrose and Hawking demonstrate that spacetime singularities are ubiquitous features of Einstein gravity [81, 82]. The utility of general relativity in describing gravitational phenomena is maintained by the cosmic censorship conjecture. This conjecture states that the undesirable spacetime singularities are always hidden inside the event horizon of black holes. The Penrose-Hawking theorem, however, does not predict that a singularity is either surrounded by an event horizon or not. If a singularity is not covered by an event horizon, in such a way that an observer at infinity can see it, it is called a naked singularity. A naked singularity can lead to some bad properties of the spacetime, for example, the stability problem, the causal problem etc. A naked singularity, however, can also be helpful to explain some mysterious astronomical phenomena and to study the structure of the spacetime if it exists [83].

Let us now consider spacetime solutions with non-trivial scalar fields $\tilde{\phi}$. We consider the Einstein-scalar (vacuum-scalar) case, for a massless, real scalar field, which is described by the action [84]

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R + 2\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi}\right). \tag{5.1}$$

A variational approach to action (5.1) will give us both the Einstein's field equations

$$R_{\mu\nu} = 8\pi \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi}, \tag{5.2}$$

and the Klein-Gordon wave equation,

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \tilde{\phi} \right) = 0. \tag{5.3}$$

According to Refs. [85, 86, 87], the line element which is considered as a rotating JNW spacetime in Einstein's frame can be written as

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2\tilde{M}r}{\Sigma}\right)^{\gamma} (dt - W d\phi)^2 + \Sigma \left(1 - \frac{2\tilde{M}r}{\Sigma}\right)^{1-\gamma} \left(\frac{dr^2}{\Delta} + d\theta^2 + \sin^2\theta d\phi^2 \right) \\ &\quad + 2W \left(dt - W d\phi\right) d\phi, \end{aligned} \tag{5.4}$$

with

$$\begin{aligned} \gamma = \frac{M}{\sqrt{M^2 + q^2}}, \quad \tilde{M} = \frac{M}{\gamma}, \quad \Sigma = r^2 + a^2 \cos^2\theta, \quad W = a \sin^2\theta, \\ \Delta = r^2 + a^2 - 2\tilde{M}r, \quad \sqrt{-g} = \Sigma^{\gamma} \sin\theta \big(\Delta - a^2 \sin^2\theta\big)^{1-\gamma}, \end{aligned} \tag{5.5}$$

where q is the integration constant and a is the rotational parameter. Turning off the rotational parameter ($a = 0$) leads us to the static JNW solution which is discovered in Ref. [79], and written in more convenient form, in (t, r, θ, ϕ) coordinates, in Ref. [88]. The Kerr solution is retrieved by setting the deformation parameter $\gamma = 1$, and the scalar field solution (5.8) vanishes. In the Kerr black hole solution, $\Delta_{\text{Kerr}} = 0$ gives the radius of the event horizon $r_{\pm}^{\text{Kerr}} = M \pm \sqrt{M^2 - a^2}$. The boundary of the ergosphere (static limit) is given by the larger root of $g_{tt} = 0$ and turns out to be $r_{\text{SL}} = M + \sqrt{M^2 - a^2 \cos^2\theta}$. For $0 < \gamma < 1$ with $q \neq 0$, the spacetime (5.4) describes a rotating naked singularity with scalar curvature which is written in Refs. [85, 86], as

$$R = \frac{2\left(\gamma^2 - 1\right)\tilde{M}^2}{\Sigma^5} \left(1 - \frac{2\tilde{M}r}{\Sigma}\right)^{\gamma - 3} \left[\Delta \big(r^2 - a^2 \cos^2\theta\big)^2 + \big(r a^2 \sin^2 2\theta\big)^2 \right], \tag{5.6}$$

and diverges on the surface-like singularity [85, 86]

$$r_* = \tilde{M} + \sqrt{\tilde{M}^2 - a^2 \cos^2\theta}. \tag{5.7}$$

The scalar field $\tilde{\phi}$ is given by

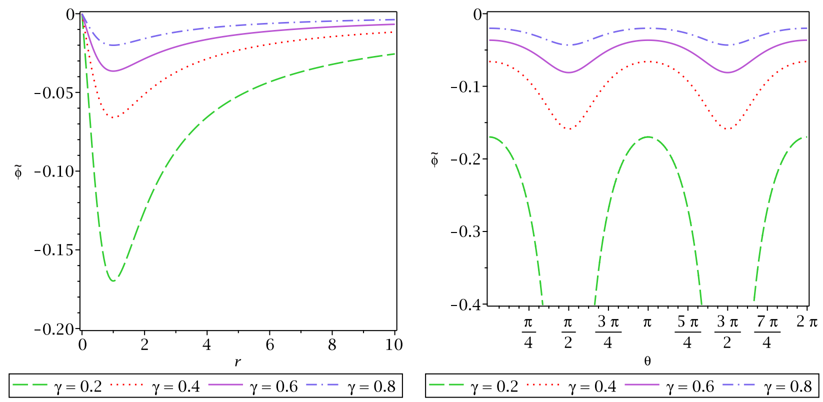
$$\tilde{\phi} = \frac{\sqrt{1 - \gamma^2}}{4} \ln \left(1 - \frac{2\tilde{M}r}{\Sigma}\right). \tag{5.8}$$

In Figs. 5.1, we notice that the scalar field (5.8) is dying as the deformation parameter γ is increasing. Note that for $\gamma = 1$ we recover the Kerr solution in which the scalar field $\tilde{\phi}$ vanishes. The contravariant metric tensor components of spacetime (5.4) can be written as

$$g^{tt} = \frac{a^2 \sin^2\theta \left[\big(\Delta - a^2 \sin^2\theta\big)^{\gamma} - 2\Sigma^{\gamma} \right] - \Sigma^{2\gamma} \big(\Delta - a^2 \sin^2\theta\big)^{1-\gamma}}{\Sigma^{\gamma} \Delta},$$

$$g^{rr} = \frac{\Delta\left(\Delta - a^2\sin^2\theta\right)^{\gamma-1}}{\Sigma^\gamma}, \quad g^{\theta\theta} = \frac{\left(\Delta - a^2\sin^2\theta\right)^{\gamma-1}}{\Sigma^\gamma},$$

$$g^{\phi\phi} = \frac{\left(\Delta - a^2\sin^2\theta\right)^{\gamma}}{\Sigma^\gamma\Delta\sin^2\theta}, \quad g^{t\phi} = g^{\phi t} = -\frac{\left[\Sigma^\gamma - \left(\Delta - a^2\sin^2\theta\right)^\gamma\right] a}{\Sigma^\gamma\Delta}. \tag{5.9}$$



(a) **(b)**

Figure 5.1: $\tilde{\phi}$ as a function of (a) r with $\theta = \pi$, (b) θ with $r = 1$, both with $a = 1$, $M = 0.1$, and several numerical values of γ.

Now we consider a massless scalar particle field Ψ outside of a rotating JNW black hole that obeys the following KG wave equation:

$$\frac{1}{\sqrt{-g}}\partial_\alpha\left(g^{\alpha\beta}\sqrt{-g}\partial_\beta\Psi\right) = 0. \tag{5.10}$$

To solve the equation above, we can, as usual, use the ansatz of the scalar field

$$\Psi\left(t, r, \theta, \phi\right) = e^{-i\omega t + im\phi}R\left(r\right)S\left(\theta\right), \tag{5.11}$$

where ω is the frequency of the wave field and m is the azimuthal harmonic index. Using line element (5.4), Eq. (5.10) becomes

$$\frac{-\omega^2\left(\Delta - a^2\sin^2\theta\right)^{1-\gamma}}{\Delta}\left\{a^2\sin^2\theta\left[\left(\Delta - a^2\sin^2\theta\right)^\gamma - 2\Sigma^\gamma\right] - \Sigma^{2\gamma}\left(\Delta - a^2\sin^2\theta\right)^{1-\gamma}\right\}$$
$$+\frac{1}{R\left(r\right)}\partial_r\left[\Delta\partial_r R\left(r\right)\right] + \frac{1}{S\left(\theta\right)\sin\theta}\partial_\theta\left[\sin\theta\partial_\theta S\left(\theta\right)\right] - \frac{m^2\left(\Delta - a^2\sin^2\theta\right)}{\Delta\sin^2\theta}$$
$$-\frac{2m\omega a\left(\Delta - a^2\sin^2\theta\right)^{1-\gamma}\left[\Sigma^\gamma - \left(\Delta - a^2\sin^2\theta\right)^\gamma\right]}{\Delta} = 0. \tag{5.12}$$

To simplify the Teukolsky master equation (5.12), we consider $\gamma = 1 - \epsilon$, where $\epsilon \ll 1$. Using series approximation and neglecting the $\mathcal{O}\left(\epsilon^2\right)$ term, Eq. (5.12) can be written in the form

$$\frac{\left[\omega\left(r^2 + a^2\right) - am\right]^2}{\Delta} - \left(a\omega\sin\theta - \frac{m}{\sin\theta}\right)^2 + \frac{1}{R\left(r\right)}\partial_r\left[\Delta\partial_r R\left(r\right)\right]$$

$$+\frac{1}{S(\theta)\sin\theta}\partial_\theta\left[\sin\theta\partial_\theta S(\theta)\right]+\frac{2\omega\epsilon\Sigma\left[\omega\left(r^2+a^2\right)-am\right]}{\Delta}\ln\left(1-\frac{2\tilde{M}r}{\Sigma}\right)=0. \tag{5.13}$$

To separate Eq. (5.13), we consider the limit $\tilde{M}\ll r$ in the far region and use the assumption that $r\ll a^2/\tilde{M}$. The radial and angular functions $R(r)$ and $S(\theta)$ are solutions to the radial equation

$$\left[\partial_r\left(\Delta\partial_r\right)+\frac{G\left(G-4\tilde{M}r\omega\epsilon\right)}{\Delta}+2ma\omega\right]R(r)=K_l R(r), \tag{5.14}$$

and angular equation

$$\left[\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right)-\frac{m^2}{\sin^2\theta}+a^2\omega^2\cos^2\theta\right]S(\theta)=K_l S(\theta), \tag{5.15}$$

respectively, where $G=\omega\left(r^2+a^2\right)-am$ and K_l is the separation constant.

5.1.2 Validity of rotating Janis-Newman-Winicour spacetime

Any scalar field which solve the Einstein equations (5.2) must also solve the scalar wave equation shown later in Eq. (5.54). Nevertheless, solving the scalar field solution (5.8) to the KG equation (5.54) yields

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\tilde{\phi}\right)=-\frac{M^2a^2\sqrt{1-\gamma^2}\left(\gamma a^2\cos^2\theta+\gamma r^2-2Mr\right)^\gamma}{\gamma^\gamma\Sigma^{\gamma+2}\left(2Mr-\gamma a^2\cos^2\theta-\gamma r^2\right)^3}$$
$$\times\left(\gamma a^4\cos^6\theta-\gamma a^4\cos^4\theta-2\gamma a^2r^2\cos^4\theta-2\gamma a^2r^2\cos^2\theta-3\gamma r^4\cos^2\theta+8Mr^3\cos^2\theta-\gamma r^4\right). \tag{5.16}$$

This result indicates that the scalar field solution (5.8) with spacetime (5.4) is not a solution of the scalar wave equation (5.54).

Next, we shall analyze the Ricci tensor component of metric (5.4) and compare it with Eq. (5.2). If Eq. (5.8) is truly the correct solution, then plugging it to Eq. (5.2) should yield

$$\begin{aligned}
R_{rr} &= -\frac{2M^2(a\cos\theta-r)^2(a\cos\theta+r)^2(\gamma-1)(\gamma+1)}{\left(2Mr-\gamma a^2\cos^2\theta-\gamma r^2\right)^2\Sigma^2},\\
R_{\theta r} &= R_{r\theta}=-\frac{4M^2a^2r\sin\theta\cos\theta\left(a\cos\theta-r\right)\left(a\cos\theta+r\right)(\gamma-1)(\gamma+1)}{\left(2Mr-\gamma a^2\cos^2\theta-\gamma r^2\right)^2\Sigma^2},\\
R_{\theta\theta} &= -\frac{8M^2a^4r^2\sin^2\theta\cos^2\theta(\gamma-1)(\gamma+1)}{\left(2Mr-\gamma a^2\cos^2\theta-\gamma r^2\right)^2\Sigma^2},
\end{aligned} \tag{5.17}$$

where the other components of $R_{\mu\nu}$ are zero. For the sake of simplicity, we will perform the calculation in the $\theta=\pi/2$ plane. Hence, Eq. (5.17) becomes

$$R_{rr}^{(\theta=\pi/2)}=-\frac{2M^2(\gamma-1)(\gamma+1)}{\left(2Mr-\gamma r^2\right)^2r^2}, \tag{5.18}$$

with $R_{\theta r}^{(\theta=\pi/2)}=R_{r\theta}^{(\theta=\pi/2)}=R_{\theta\theta}^{(\theta=\pi/2)}=0$. Solving the Ricci tensor directly from spacetime (5.4) gives

$$R_{rr}^{(\theta=\pi/2)}=\frac{1}{r^2(2M-\gamma r)^2\left(2Mr-\gamma a^2-\gamma r^2\right)}$$
$$\times\left(2M^2\gamma^3r^2+2M\gamma^3ra^2-\gamma^3r^2a^2-4M^3\gamma^2r-2M^2\gamma^2a^2+2M\gamma^2ra^2-2M^2\gamma a^2-2M^2\gamma r^2+4M^3r\right), \tag{5.19}$$

with other non-zero $R_{tt}^{(\theta=\pi/2)}$, $R_{t\phi}^{(\theta=\pi/2)}$, $R_{\phi t}^{(\theta=\pi/2)}$, and $R_{\phi\phi}^{(\theta=\pi/2)}$ components. We can see that Eq. (5.18) is not equal to Eq. (5.19). Hence, the spacetime (5.4) does not satisfy Eq. (5.2) with Eq. (5.8).

Next, we shall analyze the scalar curvature of spacetime (5.4). For $0 < \gamma < 1$ with $q \neq 0$, the spacetime (5.4) describes a rotating naked singularity with scalar curvature (5.6) and Refs. [85, 86] claim that the scalar curvature (5.6) diverges on the surface-like singularity (5.7). However, plugging Eq. (5.7) into Eq. (5.6) makes the scalar curvature vanish which contradicts their statement.

At this point, we should point out that metric (5.4) is in the Einstein's frame. The original metric in Jordan's frame was first derived in Ref. [80]. In this paper, Krori and Bhattacharjee applied the method from Newman and Janis which was first used to generate a Kerr solution from a Schwarzschild solution by using a complex coordinate transformation, and they obtained a Kerr-like solution in BD theory from the original BD metric in Ref. [89]. Unfortunately, the results of by Krori and Bhattacharjee are erroneous. The metric does not satisfy the field equation in BD theory [90, 91, 92]. Nevertheless, the Kerr-like solution of the gravitational field equation for a minimally coupled scalar field does exist in BD theory. The BD theory of gravitation is a theory where the gravitational interaction is carried by a scalar field and also a tensor field in general relativity.

5.1.3 Scalar scattering off a Brans-Dicke-Kerr black hole

In the BD theory of gravity, apart from the metric tensor, there is a scalar field which changes the effective gravitational constant from one place to another with time. The corresponding effective action in this theory reads [90]

$$S_{\mathrm{BD}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\tilde{\psi}R - \frac{\eta}{\tilde{\psi}}\partial_\mu\tilde{\psi}\partial^\mu\tilde{\psi} - V\left(\tilde{\psi}\right) + \mathcal{L}_{\mathrm{M}}\right], \tag{5.20}$$

where $\mathcal{L}_{\mathrm{M}}$ is the matter Lagrangian, η is the dimensionless BD parameter, and $V\left(\tilde{\psi}\right)$ is the scalar field potential. In our case, we consider the scalar field potential $V\left(\tilde{\psi}\right)$ and the matter Lagrangian $\mathcal{L}_{\mathrm{M}}$ to be zero. We also consider a Kerr-like metric in BD theory with a scalar field $\tilde{\psi}$ which is singular on the event horizon.

In the Einstein frame[1], varying the action (5.20) with respect to the metric tensor gives

$$R_{\mu\nu} = 8\pi\partial_\mu\tilde{\psi}\partial_\nu\tilde{\psi}, \tag{5.21}$$

and varying the action (5.20) with respect to the scalar field gives

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\tilde{\psi}\right) = 0. \tag{5.22}$$

Obtaining a stationary axisymmetric solution in BD theory is not a straightforward task. In Ref. [93], Tiwari and Nayak generate a stationary axisymmetric solution in vacuum BD theory from the Kerr solution

[1]Obtained by using the conformal transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, with $\Omega = \sqrt{G\psi}$, where $G = 1/\tilde{\psi}$ is Newton's gravitational constant written in terms of the scalar field $\tilde{\psi}$.

in Einstein's theory for a vacuum, which we will refer to as the BDK solution given by

$$\begin{aligned} ds^2 &= -\left(\frac{\Delta_{\mathrm{BDK}}-a^2\sin^2\theta}{\Sigma_{\mathrm{BDK}}}\right)dt^2-\frac{2a\sin^2\theta\left(r^2+a^2-\Delta_{\mathrm{BDK}}\right)}{\Sigma_{\mathrm{BDK}}}dtd\phi \\ &\quad+\left[\frac{\left(r^2+a^2\right)^2-\Delta_{\mathrm{BDK}}a^2\sin^2\theta}{\Sigma_{\mathrm{BDK}}}\right]\sin^2\theta d\phi^2 \\ &\quad+{\Delta_{\mathrm{BDK}}}^{4/2\eta+3}\sin^{8/2\eta+3}\theta\left(\frac{\Sigma_{\mathrm{BDK}}}{\Delta_{\mathrm{BDK}}}dr^2+\Sigma_{\mathrm{BDK}}d\theta^2\right), \end{aligned} \tag{5.23}$$

where $\Sigma_{\mathrm{BDK}}=r^2+a^2\cos^2\theta$, $\Delta_{\mathrm{BDK}}=r^2-2Mr+a^2$, with M and a denoting the mass and angular momentum per units mass, respectively. The associated scalar field is given by

$$\tilde{\psi}=\frac{1}{2\sqrt{\pi}\sqrt{2\eta+3}}\ln\left(\Delta_{\mathrm{BDK}}\sin^2\theta\right). \tag{5.24}$$

Note that the BDK metric reduces to a Kerr solution in Einstein's theory for a vacuum when the dimensionless BDK parameter $\eta\to\infty$. BDK spacetime has the same curvature singularity $\Sigma_{\mathrm{BDK}}=0$ as for the Kerr metric. For $\eta<-3/2$ or $\eta>-1/2$, BDK spacetime has the same horizon, $r_\pm=M\pm\sqrt{M^2-a^2}$, as Kerr spacetime. The Kretchmann scalar $K=R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ is finite for $-5/2\leq\eta<-3/2$ and vanishes on the horizon $r_\pm$. The scalar field $\tilde{\psi}\left(r_\pm\right)\to\infty$ so that the effective Newtonian constant tends to zero on the horizon. For $-1/2<\eta<\infty$ and $\eta<-5/2$ the Kretchmann scalar becomes infinite on outer horizon r_+. In this case, BDK spacetime has a second curvature singularity on the horizon. For $-3/2<\eta\leq 1/2$, the BDK metric has no horizon, so that the curvature singularity Σ_{BDK} is a naked singularity. To clarify, for the range $-5/2\leq\eta<-3/2$, the metric (5.23) represents a rotating black hole in BD theory and has the same horizon and curvature singularity as for Kerr spacetime. Additionally, for the range $-5/2\leq\eta<-3/2$, the scalar field in the Einstein frame (5.24) is imaginary. Moreover, setting the rotational parameter $a=0$, the BDK metric reduces to a static axisymmetric metric which can be called the Brans-Dicke-Schwarzschild metric which is not a spherically symmetric solution.

Again, we consider a massless scalar particle field Ψ outside of a BDK black hole that obeys the following KG wave equation:

$$\frac{1}{\sqrt{-g}}\partial_\alpha\left(g^{\alpha\beta}\sqrt{-g}\partial_\beta\Psi\right)=0. \tag{5.25}$$

To solve the equation above, as usual we can use the ansatz of the scalar field

$$\Psi\left(t,r,\theta,\phi\right)=e^{-i\omega t+im\phi}R\left(r\right)S\left(\theta\right), \tag{5.26}$$

where ω is the frequency of the wave field and m is the azimuthal harmonic index. Using line element (5.23), Eq. (5.25) becomes

$$\begin{gathered} {\Delta_{\mathrm{BDK}}}^{4/2\eta+3}\sin^{8/2\eta+3}\theta\left\{\frac{\left[\omega\left(r^2+a^2\right)-am\right]^2}{\Delta_{\mathrm{BDK}}}-\left(a\omega\sin\theta-\frac{m}{\sin\theta}\right)^2\right\} \\ \frac{1}{R\left(r\right)}\partial_r\left[\Delta_{\mathrm{BDK}}\partial_rR\left(r\right)\right]+\frac{1}{S\left(\theta\right)\sin\theta}\partial_\theta\left[\sin\theta\partial_\theta S\left(\theta\right)\right]=0. \end{gathered} \tag{5.27}$$

To simplify the Teukolsky master equation (5.27), we consider $\eta = 1/\sigma$, where $\sigma \ll 1$. Using a series approximation and neglecting the $\mathcal{O}\left(\sigma^2\right)$ term, Eq. (5.27) can be written in the form

$$\left\{\frac{\left[\omega\left(r^2+a^2\right)-am\right]^2}{\Delta_{\mathrm{BDK}}}-\left(a\omega\sin\theta-\frac{m}{\sin\theta}\right)^2\right\}\left[1+4\sigma\ln\left(\sin\theta\right)+2\sigma\ln\Delta_{\mathrm{BDK}}\right]$$
$$+\frac{1}{R\left(r\right)}\partial_r\left[\Delta_{\mathrm{BDK}}\partial_r R\left(r\right)\right]+\frac{1}{S\left(\theta\right)\sin\theta}\partial_\theta\left[\sin\theta\partial_\theta S\left(\theta\right)\right]=0. \tag{5.28}$$

As we can see the Teukolsky master equation (5.28) cannot be separated. Thus we cannot construct the low frequency radial wave equation to compare it to the Casimir eigen-equation in $SL(2,R)_L\times SL(2,R)_R$ algebra. In the limit of $\sigma\to 0$, however, Eq. (5.28) is separable and angular and radial KG equations (4.35)–(4.36) for Kerr black holes are recovered. We would like to note that Eq. (5.28) eliminates the possibility of any holography aspect for BDK black holes.

5.2 Hidden conformal symmetry of black holes in $f(T)$ gravity

5.2.1 Rotating charged AdS black holes

In order to obtain the rotating solution of spacetime (2.160), we need to add the angular momentum to the stationary solution by applying the coordinate transformations *locally*

$$\tilde{\phi}=-\Xi\phi+\frac{\Omega}{l^2}t,\quad \tilde{t}=\Xi t-\Omega\phi. \tag{5.29}$$

Thus, the 4D rotating charged AdS black hole solution in quadratic $f(T)$ gravity is given by

$$ds^2=-A(r)\Big(\Xi d\tilde{t}-\Omega d\tilde{\phi}\Big)^2+\frac{dr^2}{B(r)}+\frac{r^2}{l^4}\Big(\Omega d\tilde{t}-\Xi l^2 d\tilde{\phi}\Big)^2+\frac{r^2}{l^2}dz^2, \tag{5.30}$$

where the range of coordinates are given by $-\infty<t<\infty$, $-\infty<z<\infty$, $0\le r<\infty$, and $0\le\phi<2\pi$. In metric (5.30), M, Q, and Ω are the mass parameter, the charge parameter, and the rotation parameter, respectively and $\Xi=\sqrt{1+\frac{\Omega^2}{l^2}}$. The parameter α cannot be zero, since in that case the cosmological constant Λ, and the metric functions $A(r)$ and $B(r)$ become singular. Therefore, the charged rotating solutions (5.30) do not correspond to any known solutions in GR or TEGR, since in the limit of $\alpha\to 0$, the tetrad or the metric are not well-defined. The form of the tetrad fields of spacetime (5.30) with the rotation parameter included is given by

$$e_1^{\mu}=\left(\Xi\sqrt{A(r)},0,-\Omega\sqrt{A(r)},0\right),\quad e_2^{\mu}=\left(0,\frac{1}{\sqrt{B(r)}},0,0\right),\quad e_3^{\mu}=\left(\frac{\Omega r}{l^2},0,-\Xi r,0\right),\quad e_4^{\mu}=\left(0,0,0,r\right). \tag{5.31}$$

The gauge potential one-form $\tilde{\Phi}$ is given by

$$\tilde{\Phi}(r)=-\Phi(r)\left(\Omega d\phi-\Xi dt\right), \tag{5.32}$$

where $\Phi(r)=\frac{Q}{r}+\frac{Q^2\sqrt{6|\alpha|}}{3r^3}$. We note that the torsion scalar T, for the black hole solution (5.30), is given by

$$T(r)=\frac{4A'(r)B(r)}{rA(r)}+\frac{2B(r)}{r^2}, \tag{5.33}$$

where $A(r)$ and $B(r)$ are defined in Eq. (2.159).

We notice that in setting the rotational parameter $\Omega = 0$, we find the same static charged black hole configuration as in Ref. [52]. Moreover, turning off the mass and charge parameters M and Q, the metric (5.30) reduces to, the 4D AdS metric in a bizarre coordinate system.

The horizons of the black holes are the positive roots of $A(r) = 0$, among which the outer one is denoted by r_+. In Fig. 5.2, we find the real positive values of inner horizon radius $r_- = 0.034$ and the outer horizon radius $r_+ = 2.079$ for $|\alpha| = 0.05$. We also find that the radius of both horizons increases as $|\alpha|$ increases. The other four horizons are found to be imaginary and they do not correspond to a physical situation. The non-vanishing components of the contravariant metric tensor are given by

$$\begin{aligned} g^{tt} &= \frac{l^4\left(A(r)\Omega^2 - r^2\Xi^2\right)}{A(r)r^2\left(\Xi^2 l^2 - \Omega^2\right)^2}, \quad g^{rr} = B(r), \quad g^{zz} = \frac{l^2}{r^2}, \\ g^{\phi\phi} &= \frac{A(r)\Xi^2 l^4 - r^2\Omega^2}{A(r)r^2\left(\Xi^2 l^2 - \Omega^2\right)^2}, \quad g^{t\phi} = g^{\phi t} = \frac{\Xi\Omega l^2\left(A(r)l^2 - r^2\right)}{A(r)r^2\left(\Xi^2 l^2 - \Omega^2\right)^2}, \end{aligned} \tag{5.34}$$

and the determinant of the metric is

$$\sqrt{-g} = \sqrt{\frac{A(r)}{B(r)}}\frac{r^2\left(\Xi^2 l^2 - \Omega^2\right)}{l^3}. \tag{5.35}$$

We find the area of the outer horizon $\mathscr{A}$, by setting $dt = dr = 0$ in the metric (5.30), and find

$$\mathscr{A} = \int_0^{2\pi} d\phi \int_0^L dz \sqrt{-g|_{dt=dr=0}} = \frac{2\pi r_+^2 \Xi L}{l}. \tag{5.36}$$

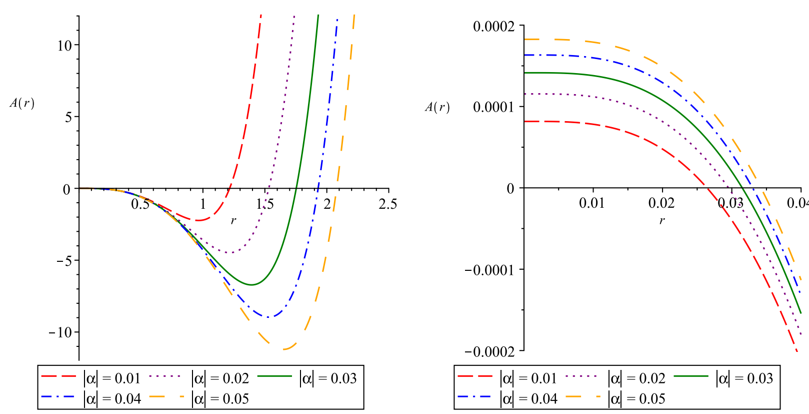


Figure 5.2: $A(r)$ as a function of r with $M = 5,\ Q = 0.1,$ and several numerical values of $|\alpha|$.

5.2.2 The first law of black hole thermodynamics

In this section, we review the black hole thermodynamics in $f(T)$ gravity [94]. In this theory, the first law of black hole thermodynamics:

$$\delta \mathcal{Q} = \tau \delta S, \tag{5.37}$$

where $\delta\mathcal{Q}$ and δS are the heat flux and the entropy change, respectively, is violated. The violation causes some degrees of freedom in $f(T)$ theory to experience a difference between the effective metric and the background metric. In other words, they observe a different black hole horizon and temperature from that observed by matter fields with the local Lorentz invariance. Let us first recall the field equations of $f(T)$ gravity with a general source of matter interaction and cosmological constant:

$$H_{\mu\nu} \equiv S_{\nu\mu\rho}\nabla^{\rho} f''(T) + \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) f'(T) + \frac{1}{2}g_{\mu\nu}\left(f(T) - f'(T)T + 2\Lambda\right) = -8\pi\mathcal{T}_{\text{matter}\,\mu\nu}, \tag{5.38}$$

with its anti-symmetric part

$$H_{[\mu\nu]} = f''(T)S_{[\nu\mu]\rho}\nabla^{\rho}T = 0, \tag{5.39}$$

where $\mathcal{T}_{\text{matter}\,\mu\nu}$ is the energy-momentum tensor of the matter. Note that when we set $f(T) = T$, we recover the Einstein equations with cosmological constant in general relativity,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = -8\pi\mathcal{T}_{\text{matter}\,\mu\nu}, \tag{5.40}$$

which verified TEGR.

The Hawking temperature is independent of the dynamics of gravity. Thus, the Hawking temperature $\tau = \kappa/2\pi$ in $f(T)$ gravity, where κ is the surface gravity, is the same as the one in Einstein's gravity. On the contrary, the entropy of black holes is associated with the dynamics of gravity. Let us discuss the first law of black hole thermodynamics in $f(T)$ gravity with the usual metric $g_{\mu\nu}$ which satisfies the Killing equation

$$\xi^{\alpha}\nabla_{\alpha}g_{\mu\nu} + \nabla_{\mu}\xi^{\alpha}g_{\alpha\nu} + \nabla_{\nu}\xi^{\alpha}g_{\alpha\mu} = 0. \tag{5.41}$$

First, we consider the heat flux δQ along a Killing vector ξ^{μ}:

$$\delta\mathcal{Q} = \int_H \mathcal{T}^{\text{em}}{}_{\mu\nu}\xi^{\mu}k^{\nu}\, d\mathscr{A}\, d\lambda, \tag{5.42}$$

where H is the black hole horizon, $\mathscr{A}$ is the surface area of the black hole, λ is the affine parameter, $k^{\mu} = dx^{\mu}/d\lambda$ is the tangent vector to H. Substituting the field equation (5.38) into Eq. (5.42), we find

$$\begin{aligned}
\delta\mathcal{Q} &= \frac{1}{8\pi}\int_H \left(f'(T)R_{\mu\nu} + S_{\nu\mu\rho}\nabla^{\rho} f'(T)\right)\xi^{\mu}k^{\nu}\, d\mathscr{A}\, d\lambda \\
&= \frac{1}{8\pi}\int_H k^{\nu}\left(f'(T)R_{\mu\nu}\xi^{\mu} + \xi^{\mu}S_{\nu\mu\rho}\nabla^{\rho} f'(T)\right) d\mathscr{A}\, d\lambda \\
&= \frac{1}{8\pi}\int_H k^{\nu}\left(f'(T)\nabla_{\mu}\nabla_{\nu}\xi^{\mu} + \xi^{\mu}S_{\nu\mu\rho}\nabla^{\rho} f'(T)\right) d\mathscr{A}\, d\lambda \\
&= \frac{1}{8\pi}\int_H k^{\nu}\left[\nabla^{\mu}\left(f'(T)\nabla_{\nu}\xi_{\mu}\right) - \left(\nabla^{\mu} f'(T)\right)\nabla_{\nu}\xi_{\mu} + \xi^{\mu}S_{\nu\mu\rho}\nabla^{\rho} f'(T)\right] d\mathscr{A}\, d\lambda \\
&= \frac{1}{8\pi}\int_H k^{\nu}\left[l^{\mu}\left(f'(T)\nabla_{\nu}\xi_{\mu}\right) - \left(l^{\mu} f'(T)\right)\nabla_{\nu}\xi_{\mu} + \xi^{\mu}S_{\nu\mu\rho}\nabla^{\rho} f'(T)\right] d\mathscr{A}\, d\lambda \\
&= \frac{\kappa}{2\pi}\left(\frac{f'(T)\, d\mathscr{A}}{4}\right)\Big|_0^{d\lambda} + \frac{1}{8\pi}\int_H k^{\nu}\nabla^{\mu} f'(T)\left(\xi^{\rho}S_{\rho\nu\mu} - \nabla_{\nu}\xi_{\mu}\right) d\mathscr{A}\, d\lambda.
\end{aligned} \tag{5.43}$$

In the above derivation, we use following formulas:

$$k^{\mu}\xi_{\nu}=0,\quad k^{\mu}k_{\mu}=0,\quad l^{\mu}l_{\mu}=0,\quad k^{\mu}l_{\mu}=-1,$$

$$R_{\mu\nu}\xi^{\mu}=\nabla_{\mu}\nabla_{\nu}\xi^{\mu},\quad \xi^{\mu}\nabla_{\mu}R=0,\quad k^{\mu}l^{\nu}\nabla_{\mu}\xi_{\nu}=\kappa,\quad \tau=\frac{\kappa}{2\pi},\quad \frac{d\kappa}{d\lambda}=0. \tag{5.44}$$

Note that in Eq. (5.43), we assume $\xi^{\mu}\sim k^{\mu}$ on the horizon surface, therefore $\xi^{\mu}k^{\nu}S_{\mu\nu\rho}=\xi^{\nu}k^{\mu}S_{\mu\nu\rho}$. Considering $\xi^{\mu}\sim k^{\mu}$, the anti-symmetric part (5.39) does not contribute to the heat flux equation (5.43). The first term of Eq. (5.43) can be explained as the first law of black hole thermodynamics:

$$\frac{\kappa}{2\pi}\left(\frac{f'(T)\,d\mathscr{A}}{4}\right)\Big|_{0}^{d\lambda}=\tau\delta S. \tag{5.45}$$

However, the second term is not equal to zero. This term maybe regarded as a contribution to the intrinsic entropy production:

$$\frac{1}{8\pi}\int k^{\nu}\nabla^{\mu}f'(T)\left(\xi^{\rho}S_{\rho\nu\mu}-\nabla_{\nu}\xi_{\mu}\right)d\mathscr{A}\,d\lambda=-\tau\delta S_{i}. \tag{5.46}$$

Next, we will review the proof of why the intrinsic entropy production cannot vanish. First, in Eq. (5.46), $k^{\nu}\nabla_{\nu}\xi_{\mu}$ is invariant under Lorentz transformation while $k^{\nu}\xi^{\rho}S_{\rho\nu\mu}$ is not. Thus, they cannot cancel each other out. Second, we assume that the second term of Eq. (5.43) can be written as

$$\int_{H}k^{\nu}\nabla^{\mu}B_{[\nu\mu]}d\mathscr{A}\,d\lambda, \tag{5.47}$$

for an arbitrary $f'(T)$, where $B_{[\nu\mu]}=f'(T)\left(\xi^{\rho}S_{\rho\nu\mu}-\nabla_{\nu}\xi_{\mu}\right)$. Applying ∇^{μ} to $B_{[\nu\mu]}$,

$$\nabla^{\mu}B_{[\nu\mu]}=\nabla^{\mu}f'(T)\left(\xi^{\rho}S_{\rho\nu\mu}-\nabla_{\nu}\xi_{\mu}\right), \tag{5.48}$$

but we have the identity in which $\nabla^{\nu}\nabla^{\mu}B_{[\nu\mu]}=R^{\mu\nu}B_{[\nu\mu]}=0$. Thus, we can obtain

$$\nabla^{\nu}\nabla^{\mu}B_{[\nu\mu]}=\nabla^{\mu}f'(T)\,\nabla^{\nu}\left(\xi^{\rho}S_{\rho\nu\mu}-\nabla_{\nu}\xi_{\mu}\right)=0. \tag{5.49}$$

Again, we see that $\nabla^{\nu}\nabla_{\nu}\xi_{\mu}$ is invariant under Lorentz transformation, but not $\nabla^{\nu}\left(\xi^{\rho}S_{\rho\nu\mu}\right)$. Hence, the second term of Eq. (5.43) cannot be written in the form (5.47). For now, we conclude that the first law of thermodynamics in $f(T)$ gravity is violated. However, there might be some cases in which the first law of thermodynamics for the $f(T)$ black holes can be recovered.

According to Miao et al. [94], the entropy and the entropy production must be positive. Therefore, there are two constraints for $f(T)$ gravity:

$$f'(T)>0,\quad f''(T)k^{\nu}\nabla^{\mu}T(\xi^{\rho}S_{\rho\nu\mu}-\nabla_{\nu}\xi_{\mu})\leq 0. \tag{5.50}$$

In the paper [95], Muller et al. conduct an experiment to examine the local Lorentz invariance in the gravitational sector. It turns out that they found a small violation of local Lorentz invariance. Consistently with their experiment, the violation of the local Lorentz invariance in $f(T)$ gravity theory must be small as well. The parameter $f''(T)$ can be used to signify the violation of the local Lorentz invariance. For this reason, we require that $f''(T) \ll 1$ and the entropy production term in Eq. (5.43) can be neglected. In this approximation, we can recover the first law of black hole thermodynamics in $f(T)$ gravity and the entropy becomes

$$S = \frac{f'(T)\mathscr{A}}{4}. \tag{5.51}$$

In the final analysis, if we set $f'(T) = 1$, the entropy (5.51) reduces to the Bekenstein-Hawking entropy in Einstein gravity:

$$S = \frac{\mathscr{A}}{4}, \tag{5.52}$$

which confirm TEGR once again.

Using Eq. (5.36) in (5.51) and recalling $f(T) = T + \alpha T^2$, we find the entropy of black holes in Eq. (5.30), as

$$S = \frac{\pi \Xi L \left(7{r_+}^6 + 9\sqrt{6|\alpha|}Q{r_+}^4 + 18M|\alpha|{r_+}^3 - 54Q^2|\alpha|{r_+}^2 - 42\sqrt{6|\alpha|^3}Q^3\right)}{9l\left(Q\sqrt{6|\alpha|} + {r_+}^2\right)^2}. \tag{5.53}$$

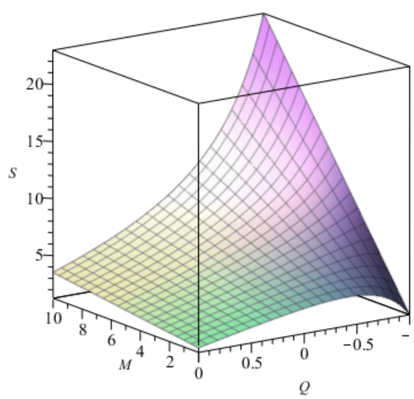
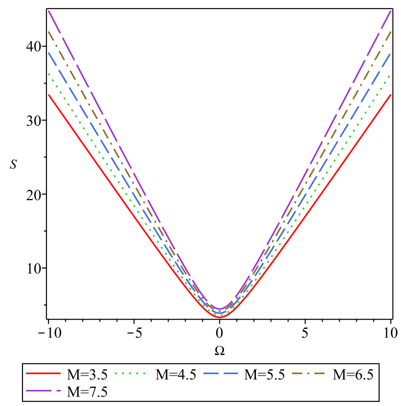
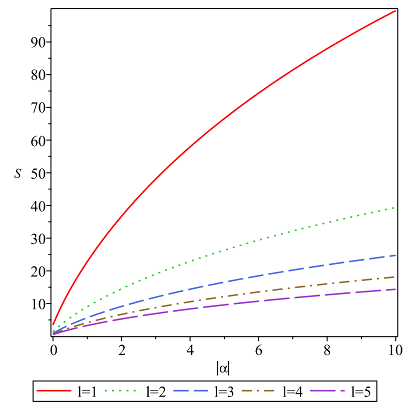


Figure 5.3: S as a function of M and Q. In these plots, we set $\Omega = 1,\ l = 1,\ |\alpha| = 0.05,\ L = 1,$ and $r_+ = 1$.



(a)



(b)

Figure 5.4: S as a function of (a) Ω with $l = 1$, $\alpha = 0.05$, and several numerical values of M. (b) $|\alpha|$ with $M = 3.5$, $\Omega = 1$, and several numerical values of l, both with $Q = 1$, $L = 1$, and $r_+ = 1$.

Figure 5.3 shows the behavior of the entropy (5.53) versus the black hole parameters M and Q. Moreover, Figs. 5.4a and 5.4b show the entropy versus Ω and $|\alpha|$, respectively.

5.2.3 Scalar scattering off a rotating charged AdS black hole

Consider a massless KG field ψ outside of a rotating charged AdS black hole (5.30) in $f(T)$ gravity that obeys the following scalar wave equation:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right) = 0. \tag{5.54}$$

To solve the above equation, as usual, we can use the ansatz of the scalar field

$$\psi\left(t, r, z, \phi\right) = e^{-i\omega t + ikz + im\phi}R\left(r\right), \tag{5.55}$$

where ω is the frequency of the scalar field, m is the azimutal harmonic index, and k is the wave number. Substituting Eqs. (5.34), (5.35), and (5.55) into Eq. (5.54), we find the radial equation

$$B(r)\frac{d^2R\left(r\right)}{dr^2} + \frac{\left(rB(r)\frac{dA(r)}{dr} + rA(r)\frac{dB(r)}{dr} + 4A(r)B(r)\right)}{2rA(r)}\frac{dR(r)}{dr} + V(r)R\left(r\right) = 0, \tag{5.56}$$

where the potential $V(r)$, is given by

$$V(r) = \frac{r^2\left(\Xi l^2\omega - \Omega m\right)^2 - A(r)l^2\left\{k^2l^4\Xi^4 + k^2\Omega^4 + l^2\left[m^2\Xi^2 - 2m\Xi\Omega\omega + \Omega^2\left(\omega^2 - 2\Xi^2k^2\right)\right]\right\}}{A(r)r^2\left(\Xi^2l^2 - \Omega^2\right)^2}. \tag{5.57}$$

We can rewrite the radial equation (5.56) as

$$\beta(r)A(r)\frac{d^2R\left(r\right)}{dr^2} + \left[\frac{\beta(r)}{2}\frac{dA(r)}{dr} + \frac{1}{2}\frac{d\left(\beta(r)A(r)\right)}{dr} + \frac{2A(r)\beta(r)}{r}\right]\frac{dR(r)}{dr} + V(r)R\left(r\right) = 0. \tag{5.58}$$

In the near-horizon region, we expand the metric function $A(r)$ using a Taylor expansion as a quadratic polynomial in $(r-r_+)$, such as

$$\begin{aligned}
A(r) &\simeq A(r_+)+(r-r_+)A'(r_+)+\frac{(r-r_+)^2}{2}A''(r_+)+\dots \\
&= 0+(r-r_+)(6r^5\Lambda-3Mr^2+3Q^2r)+\frac{(r-r_+)^2}{2}(30r^4\Lambda-6Mr+3Q^2)+\dots \\
&= a_1K(r-r_+)+K(r-r_+)^2+\dots \\
&= K(r-r_+)(r-r_++a_1)=K(r-r_+)(r-r_*) \qquad (5.59)
\end{aligned}$$

where

$$K=15{r_+}^4\Lambda-3Mr_++\frac{3Q^2}{2}, \quad r_*=r_+-\frac{2r_+\left(2{r_+}^4\Lambda-Mr_++Q^2\right)}{10{r_+}^4\Lambda-2Mr_++Q^2}. \qquad (5.60)$$

We note that r_* is not necessarily any of the black hole horizons. In the near-horizon region, we consider the low-energy limit for the scalar fields, where

$$r_+\ll\frac{1}{\omega}. \qquad (5.61)$$

Moreover, we consider a limit where the outer horizon r_+ is very close to r_*, in which

$$|r_+-r_*|\ll r_+ \text{ or } a_1\ll r_+. \qquad (5.62)$$

Substituting Eq. (5.59) into the radial Eq. (5.58) we find

$$\begin{aligned}
&\beta(r)K(r-r_+)(r-r_*)\frac{d^2R(r)}{dr^2} \\
+ &\left\{\frac{\beta(r)}{2}\frac{d\left[K(r-r_+)(r-r_*)\right]}{dr}+\frac{1}{2}\frac{d\left[\beta(r)K(r-r_+)(r-r_*)\right]}{dr}+\frac{2K(r-r_+)(r-r_*)\beta(r)}{r_+}\right\}\frac{dR(r)}{dr} \\
+ &V(r)R(r)=0. \qquad (5.63)
\end{aligned}$$

Using approximations (5.61) and (5.62), we find that the radial equation (5.63) simplifies to

$$\begin{aligned}
&\beta(r)K(r-r_+)(r-r_*)\frac{d^2R(r)}{dr^2} \\
+ &\left[\beta(r)K(r-r_*)+\beta(r)K(r-r_+)+\frac{2K(r-r_+)(r-r_++a_1)\beta(r)}{r_+}\right]\frac{dR(r)}{dr}+V(r)R(r)=0. \\
&\beta(r)K(r-r_+)(r-r_*)\frac{d^2R(r)}{dr^2} \\
+ &\left[\beta(r)K(r-r_*)+\beta(r)K(r-r_+)+\underbrace{\frac{2K(r-r_+)^2\beta(r)}{r_+}}_{\to 0}\right]\frac{dR(r)}{dr}+V(r)R(r)=0. \qquad (5.64)
\end{aligned}$$

Finally, after rearrangement, the radial equation can be written as

$$\frac{d}{dr}\left[(r-r_+)(r-r_*)\frac{d}{dr}R(r)\right]+\left[\left(\frac{r_+-r_*}{r-r_+}\right)\mathcal{A}+\left(\frac{r_+-r_*}{r-r_*}\right)\mathcal{B}+\mathcal{C}\right]R(r)=0, \qquad (5.65)$$

where the constants $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are given by

$$\mathcal{A} = \frac{\mathcal{D}m^2+\mathcal{E}m\omega}{K^2{r_+}^2{r_*}^3(\Xi^2l^2-\Omega^2)^2(r_+-r_*)^2\beta}+\frac{\mathcal{F}\omega^2}{K{r_+}^2(\Xi^2l^2-\Omega^2)^2(r_+-r_*)^2\beta}-C_1, \qquad (5.66)$$

$$\mathcal{B} = \frac{\mathcal{G}m^2+\mathcal{I}m\omega}{K^2{r_+}^3r_*(\Xi^2l^2-\Omega^2)^2(r_+-r_*)^2\beta}+\frac{\mathcal{J}\omega^2}{K{r_+}^2(\Xi^2l^2-\Omega^2)^2(r_+-r_*)^2\beta}+C_2, \qquad (5.67)$$

$$\mathcal{C} = -\frac{2m\Omega\left(\Xi l^2\omega - \Omega m/2\right)^2\left({r_+}^2 + r_+ r_* + {r_*}^2\right)}{\left(\Xi^2 l^2 - \Omega^2\right)^2 K^2 {r_+}^3 {r_*}^3 \beta}. \tag{5.68}$$

In Eqs. (5.66)–(5.67), the constants C_1 and C_2 are given by $C_1 = C_2 = k^2 l^2 / K\beta (r_+ - r_*)^2 {r_+}^2$, and

$$\mathcal{D} = \Omega^2\left({r_+}^3 + 2{r_+}^2 r_* + 3{r_*}^2 r_+\right) - l^4\Xi^2 {r_*}^3 K, \quad \mathcal{E} = 2\Omega\Xi l^2\left(Kl^2 {r_*}^3 - {r_+}^3 - 2{r_+}^2 r_* - 3r_+{r_*}^2\right),$$

$$\mathcal{F} = -l^4\Omega^2, \quad \mathcal{G} = K\Xi^2 l^4 r_+ r_* - \Omega^2\left(3{r_+}^2 + 2r_+ r_* + {r_*}^2\right),$$

$$\mathcal{I} = 2l^2\Xi\Omega\left[3{r_+}^2 - r_+ r_*\left(Kl^2 - 2\right) + {r_*}^2\right], \quad \mathcal{J} = l^4\Omega^2. \tag{5.69}$$

5.2.4 Hidden conformal symmetry

As usual, to find the existence of possible hidden symmetries, we introduce the following conformal coordinates ω^+, ω^- and y, in terms of the black hole coordinates t, r and ϕ:

$$\omega^+ = \sqrt{\frac{r - r_+}{r - r_*}} e^{2\pi T_R \phi + 2n_R t}, \quad \omega^- = \sqrt{\frac{r - r_+}{r - r_*}} e^{2\pi T_L \phi + 2n_L t}, \quad y = \sqrt{\frac{r_+ - r_*}{r - r_*}} e^{\pi(T_L + T_R)\phi + (n_L + n_R)t}, \tag{5.70}$$

where T_L, T_R, n_L and n_R are constants. We also define the sets of *local* vector fields

$$H_1 = i\partial_+, \quad H_0 = i\left(\omega^+\partial_+ + \frac{1}{2}y\partial_y\right), \quad H_{-1} = i(\omega^{+2}\partial_+ + \omega^+ y\partial_y - y^2\partial_-), \tag{5.71}$$

as well as

$$\bar{H}_1 = i\partial_-, \quad \bar{H}_0 = i\left(\omega^-\partial_- + \frac{1}{2}y\partial_y\right), \quad \bar{H}_{-1} = i(\omega^{-2}\partial_- + \omega^- y\partial_y - y^2\partial_+). \tag{5.72}$$

The vector fields (5.71)–(5.72) obey the $SL(2,R)_L \times SL(2,R)_R$ algebra, as

$$[H_0, H_{\pm 1}] = \mp i H_{\pm 1}, \quad [H_{-1}, H_1] = -2iH_0, \quad [\bar{H}_0, \bar{H}_{\pm 1}] = \mp i\bar{H}_{\pm 1}, \quad [\bar{H}_{-1}, \bar{H}_1] = -2i\bar{H}_0. \tag{5.73}$$

The quadratic Casimir operators of the $SL(2,R)_L \times SL(2,R)_R$ algebra, are given by

$$\mathcal{H}^2 = \bar{\mathcal{H}}^2 = -H_0^2 + \frac{1}{2}(H_1 H_{-1} + H_{-1}H_1) = \frac{1}{4}(y^2\partial_y^2 - y\partial_y) + y^2\partial_+\partial_-. \tag{5.74}$$

We notice that the Casimir operators (5.74), can be rewritten in terms of (t, r, ϕ) coordinates, as

$$\begin{aligned} \mathcal{H}^2 \quad = \quad & (r - r_+)(r - r_*)\frac{\partial^2}{\partial r^2} + (2r - r_+ - r_*)\frac{\partial}{\partial r} + \left(\frac{r_+ - r_*}{r - r_*}\right)\left[\left(\frac{n_L - n_R}{4\pi G}\partial_\phi - \frac{T_L - T_R}{4G}\partial_t\right)^2 + C_2\right] \\ & - \left(\frac{r_+ - r_*}{r - r_+}\right)\left[\left(\frac{n_L + n_R}{4\pi G}\partial_\phi - \frac{T_L + T_R}{4G}\partial_t\right)^2 - C_1\right], \end{aligned} \tag{5.75}$$

where $G = n_L T_R - n_R T_L$.

The Casimir operator (5.75) reproduces the radial equation (5.65), by choosing the right and left temperatures, as

$$T_R = \frac{r_+ K\left(r_+ - r_*\right)\left(\Xi^2 l^2 - \Omega^2\right)\sqrt{\beta r_+ r_* \delta}}{4\pi\delta}, \tag{5.76}$$

$$T_L = \frac{r_+ K\left(\Xi^2 l^2 - \Omega^2\right)\left[{r_+}^4 + 2{r_+}^3 r_* + 6{r_+}^2{r_*}^2 - 2{r_*}^3 r_+\left(Kl^2 - 1\right) + {r_*}^4\right]\sqrt{\beta r_+ r_* \delta}}{4\pi(r_+ + r_*)^3\delta}, \tag{5.77}$$

respectively, and

$$n_R = 0, \tag{5.78}$$

$$n_L = \frac{{r_*}^2 r_+ K\left(\Omega^2 - \Xi^2 l^2\right)\sqrt{\beta r_+ r_* \delta}}{2\Omega l^2 \Xi {(r_+ + r_*)}^3}. \tag{5.79}$$

The constant δ in Eqs. (5.76)–(5.79), is given by $\delta = 3\Omega^2 {r_+}^2 - r_+ r_* \left(K\Xi^2 l^4 - 2\Omega^2\right) + \Omega^2 {r_*}^2$. Moreover, we find two constraints for the parameters of the black hole solutions (5.30), such as

$$\frac{\Omega^2\Xi^2 l^4\left[-3{r_+}^2 + r_* r_+\left(Kl^2 - 2\right) - {r_*}^2\right]^2}{K^2\beta {r_+}^3 r_* {(r_+ - r_*)}^2 (\Xi^2 l^2 - \Omega^2)^2} = \frac{l^4\Omega^2}{{r_+}^2 K (\Xi^2 l^2 - \Omega^2)^2 {(r_+ - r_*)}^2 \beta}, \tag{5.80}$$

and

$$\frac{\Omega^2\Xi^2 l^4\left(Kl^2 {r_*}^3 - {r_+}^3 - 2r_* {r_+}^2 - 3{r_*}^2 r_+\right)^2}{K^2\beta r_+ {r_*}^5 {(r_+ - r_*)}^2 (\Omega^2 - \Xi^2 l^2)^2 \left(Kl^4\Xi^2 r_* r_+ - 3\Omega^2 {r_+}^2 - 2\Omega^2 r_* r_+ - \Omega^2 {r_*}^2\right)} = \frac{l^4\Omega^2}{{r_+}^2 K (\Xi^2 l^2 - \Omega^2)^2 {(r_+ - r_*)}^2 \beta}. \tag{5.81}$$

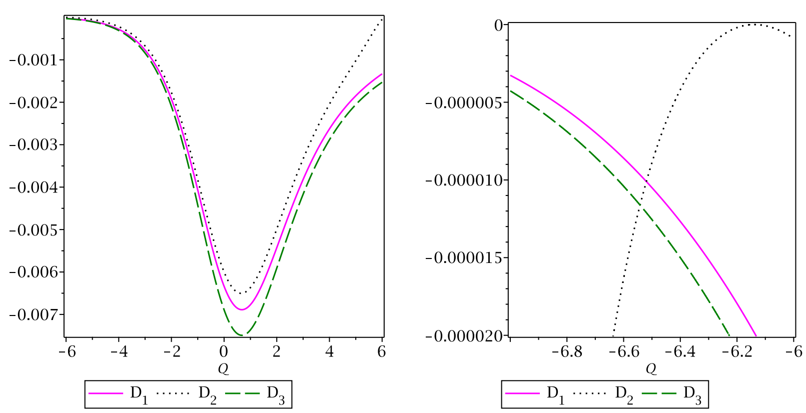


Figure 5.5: Equations (5.80) and (5.81), respectively, as a function of Q. In this plot, we set, $M = 5,\ \Omega = 1,\ l = 1,\ |\alpha| = 0.05,$ and $r_+ = 2.08$.

We define the left-hand side of Eqs. (5.80) and (5.81) as D_1 and D_2, respectively. We also define the right-hand side of Eqs. (5.80) and (5.81) as D_3. We note that Eqs. (5.80) and (5.81) restrict the black hole parameters, in accordance with the existence of real positive values for the outer horizon (and any other horizons), as the roots of the triple-quadratic algebraic equation

$$A(r) = 0, \tag{5.82}$$

where $A(r)$ is given in (2.159). In Fig. 5.5, we notice that Eqs. (5.80) and (5.81) are approximately equal when $Q = -6.5309$.

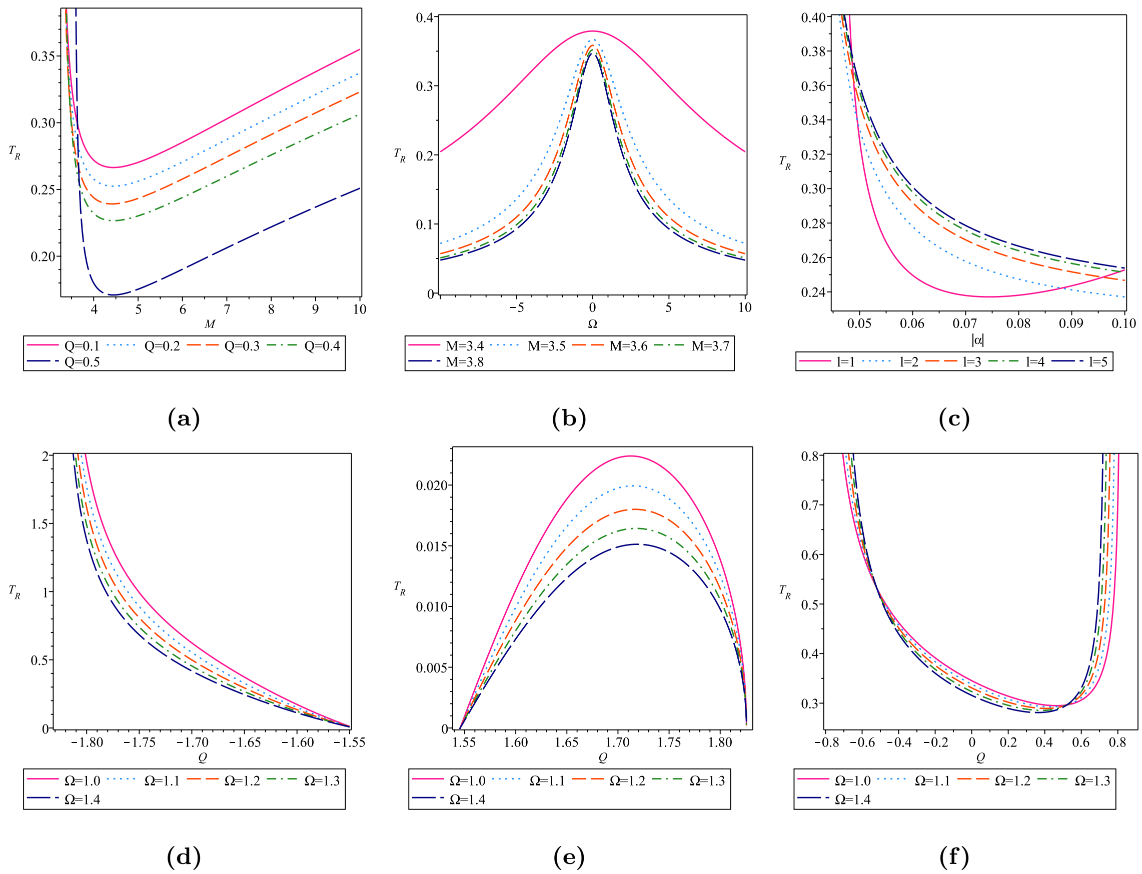


Figure 5.6: T_R as a function of (a) M, with $\Omega = 1,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of Q. (b) Ω, with $Q = 0.1,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of M. (c) $|\alpha|$, with $Q = 0.1,\ M = 3.5,\ \Omega = 1$, and several numerical values of l. (d), (e), and (f) Q, with $M = 3.5,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of Ω. In these plots, we set, $r_+ = 1$.

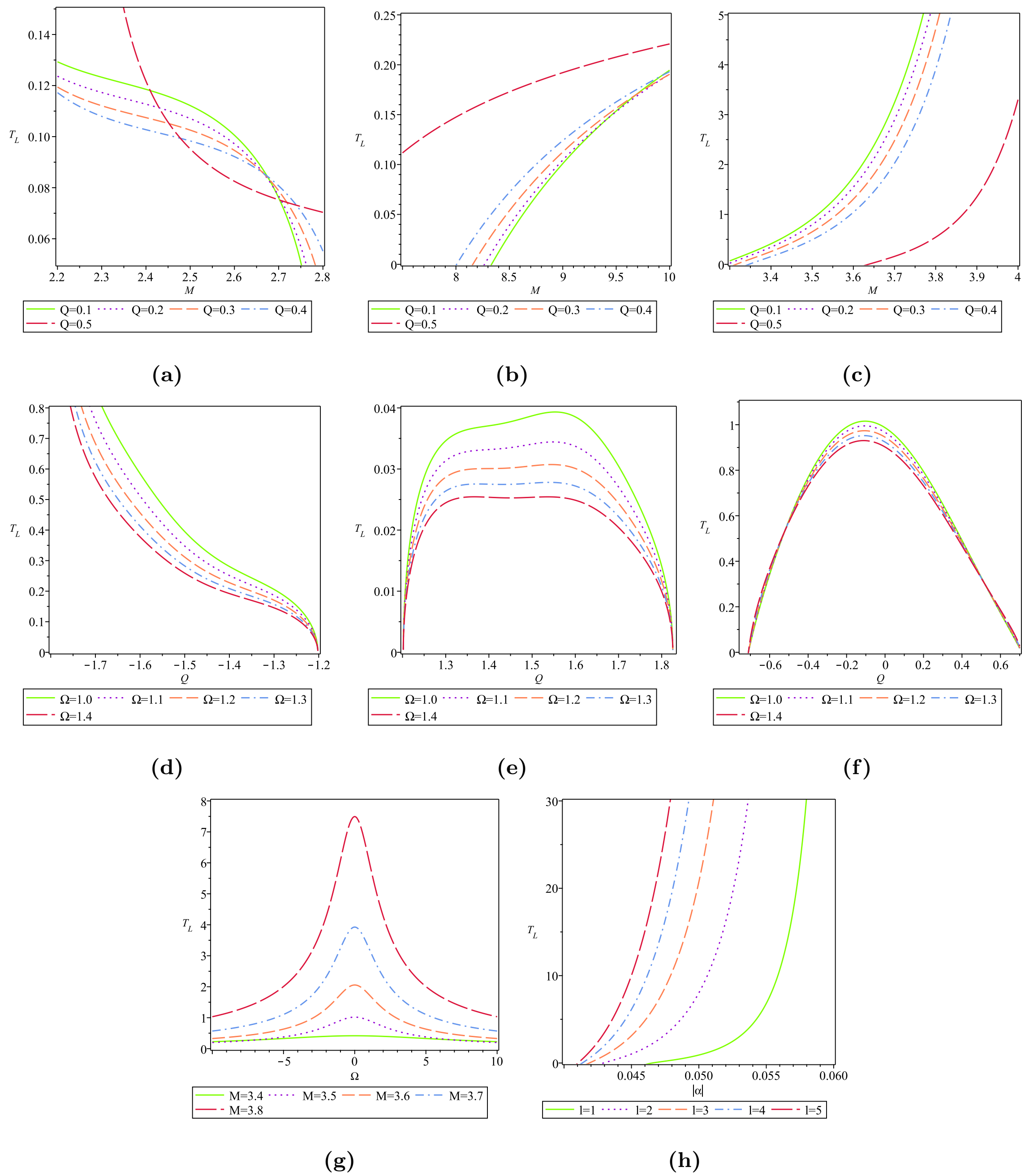


Figure 5.7: T_L as a function of (a), (b), and (c) M, with $\Omega = 1,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of Q. (d), (e), and (f) Q, with $M = 3.5,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of Ω. (g) Ω, with $Q = 0.1,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of M. (h) $|\alpha|$, with $Q = 0.1,\ \Omega = 1,\ M = 3.5$, and several numerical values of l. In these plots, we set, $r_+ = 1$.

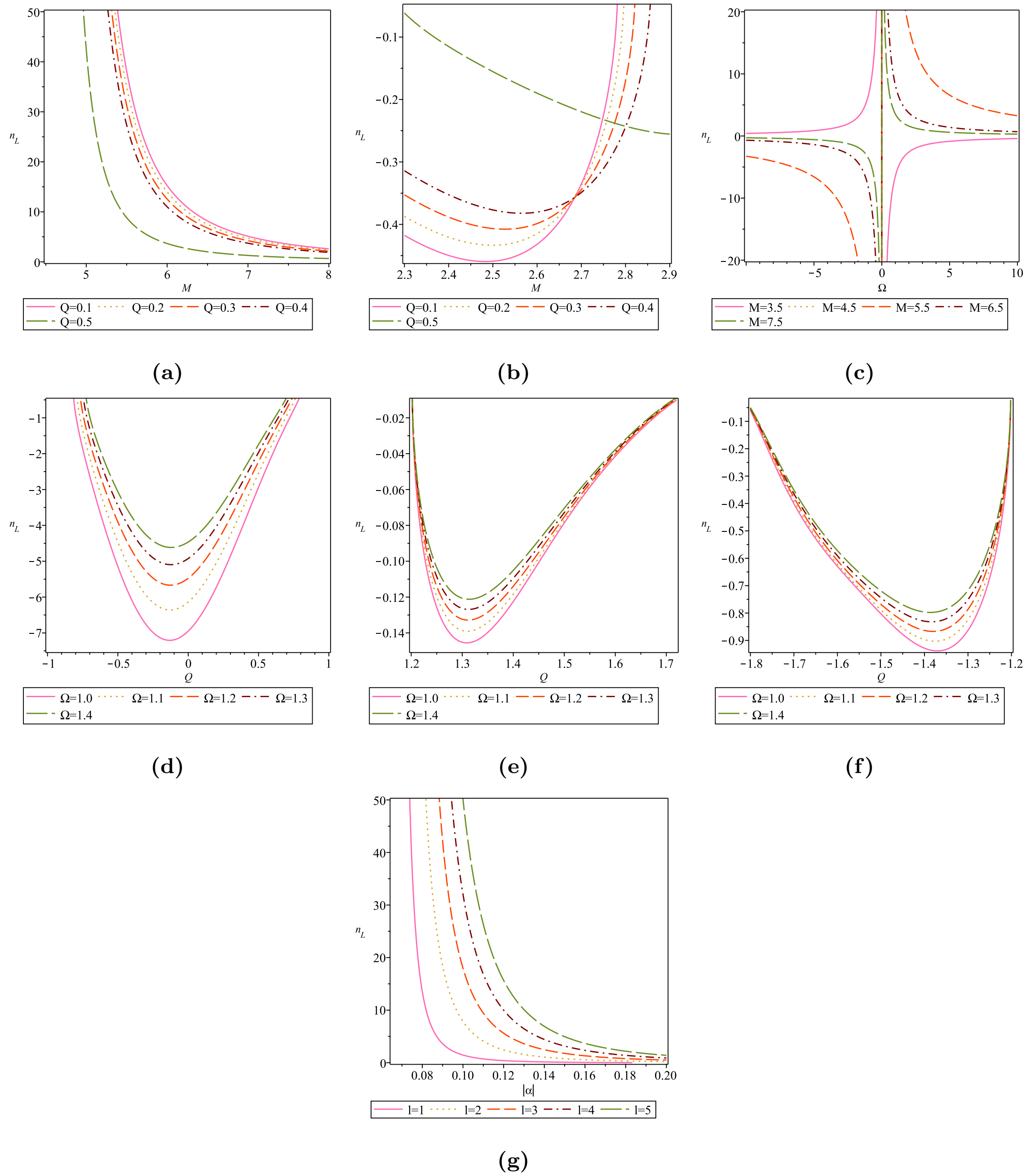
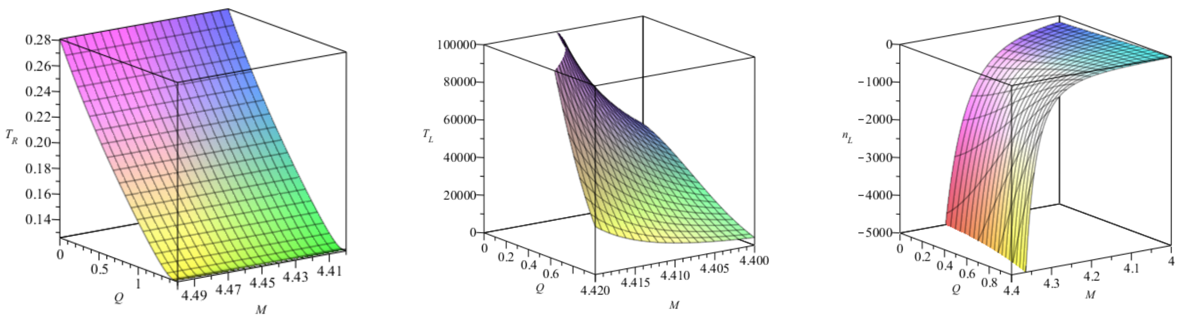


Figure 5.8: n_L as a function of (a) and (b) M, with $\Omega = 1,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of Q. (c) Ω, with $Q = 0.1,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of M. (d), (e), and (f) Q, with $M = 3.5,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of Ω. (g) $|\alpha|$, with $Q = 0.1,\ \Omega = 1,\ M = 3.5$, and several numerical values of l. In these plots, we set, $r_+ = 1$.

Figure 5.6 shows the typical behavior of the right temperature (5.76) versus different black hole parameters. Figures 5.6a, 5.6b and 5.6c show the right temperature versus M, Ω and $|\alpha|$, respectively. Moreover, Figs. 5.6d, 5.6e and 5.6f show the right temperature versus Q, for three different ranges of the electric charge, where the black hole (5.30) has real positive outer horizon.

Figure 5.7 shows the typical behavior of the left temperature (5.77) versus different black hole parameters. Figures 5.7a, 5.7b and 5.7c show the left temperature versus M, for three different ranges of the mass parameter, where the black hole (5.30) has a real positive outer horizon. Figures 5.7d, 5.7e and 5.7f show the left temperature versus Q, for three different ranges of the electric charge, where the black hole (5.30) has a real positive outer horizon. Moreover, Fig. 5.7g shows the left temperature versus Ω, for several values of the mass parameter, where the left temperature is positive definite. Finally, Fig. 5.7h shows the monotonically increasing behavior of the left temperature, versus $|\alpha|$.

Continuing, Figs. 5.8a to 5.8g show the behavior of n_L as in (5.79), versus the black hole parameters M, Ω, Q and $|\alpha|$. We note that $n_R = 0$, according to Eq. (5.78). We also plot the right temperature (5.76), the left temperature (5.77), and n_L (5.79), versus the black hole mass parameter M and the electric charge Q, in Figs. 5.9a, 5.9b, and 5.9c, respectively.



(a) (b) (c)

Figure 5.9: (a) T_R, (b) T_L, and (c) n_L, respectively, as a function of M and Q. In these plots, we set, $\Omega = 1,\ l = 1,\ |\alpha| = 0.05,\ \text{and}\ r_+ = 1$.

5.2.5 Cardy entropy

We recall the Cardy entropy formula for the dual 2D CFT with temperatures T_L and T_R

$$S_{\text{CFT}} = \frac{\pi^2}{3}\left(c_L T_L + c_R T_R\right), \tag{5.83}$$

where c_L and c_R are the corresponding central charges for the left and right sectors. The central charges can be derived from the asymptotic symmetry group of the near-horizon (near-)extremal black hole geometry. There is no derivation for the central charges of the CFT dual to the non-extremal black holes that we consider in this article. Of course, we expect that the conformal symmetry of the extremal black holes connects smoothly to those of the non-extremal black holes, for which the central charges are the same. The near-horizon extremal geometry for spacetime (5.30) has not been discovered yet and it is not a straightforward task to find it, due to the triple-quadratic behavior of the metric function $A(r)$. As a result, we turn the logic around and consider the favorite holographic situation in which the Cardy entropy (5.83) produces exactly the macroscopic entropy (5.51). Substituting Eqs. (5.53), (5.76), and (5.77) to Eq. (5.83), we find the central charges

$$c \equiv c_L = c_R = \frac{12\Xi\delta L\varpi(r_+ + r_*)^3}{lK{r_+}^2\left(\Xi^2 l^2 - \Omega^2\right)\left({r_+}^3 + 2{r_+}^2 r_* + 3r_+{r_*}^2 - l^2 K{r_*}^3\right)\left(Q\sqrt{6\,|\alpha|} + {r_+}^2\right)^2\sqrt{\beta r_+ r_* \delta}}, \quad (5.84)$$

where $\varpi = {r_+}^2\left({r_+}^2 Q\sqrt{6\,|\alpha|}/2 + 7{r_+}^4/18 + M\,|\alpha|\,r_+ - 3Q^2\,|\alpha|\right) - 7\sqrt{6|\alpha|^3}Q^3/3$. We note that we only consider CFTs in which the left and right central charges are equal, $c \equiv c_L = c_R$ [75, 96]. In Figs. 5.11–5.10 we plot the behavior of the central charges (5.84) of the dual CFT, as a function of M, Ω, Q, and $|\alpha|$.

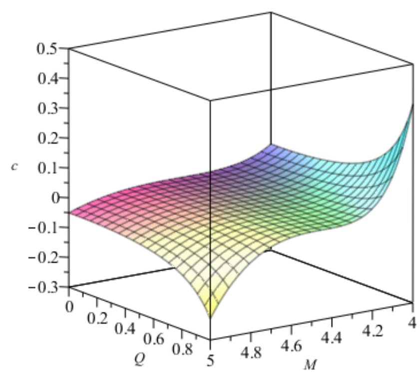


Figure 5.10: c as a function of M and Q. In these plots, we set, $\Omega = 1,\ l = 1,\ |\alpha| = 0.05,\ L = 1,$ and $r_+ = 1$.

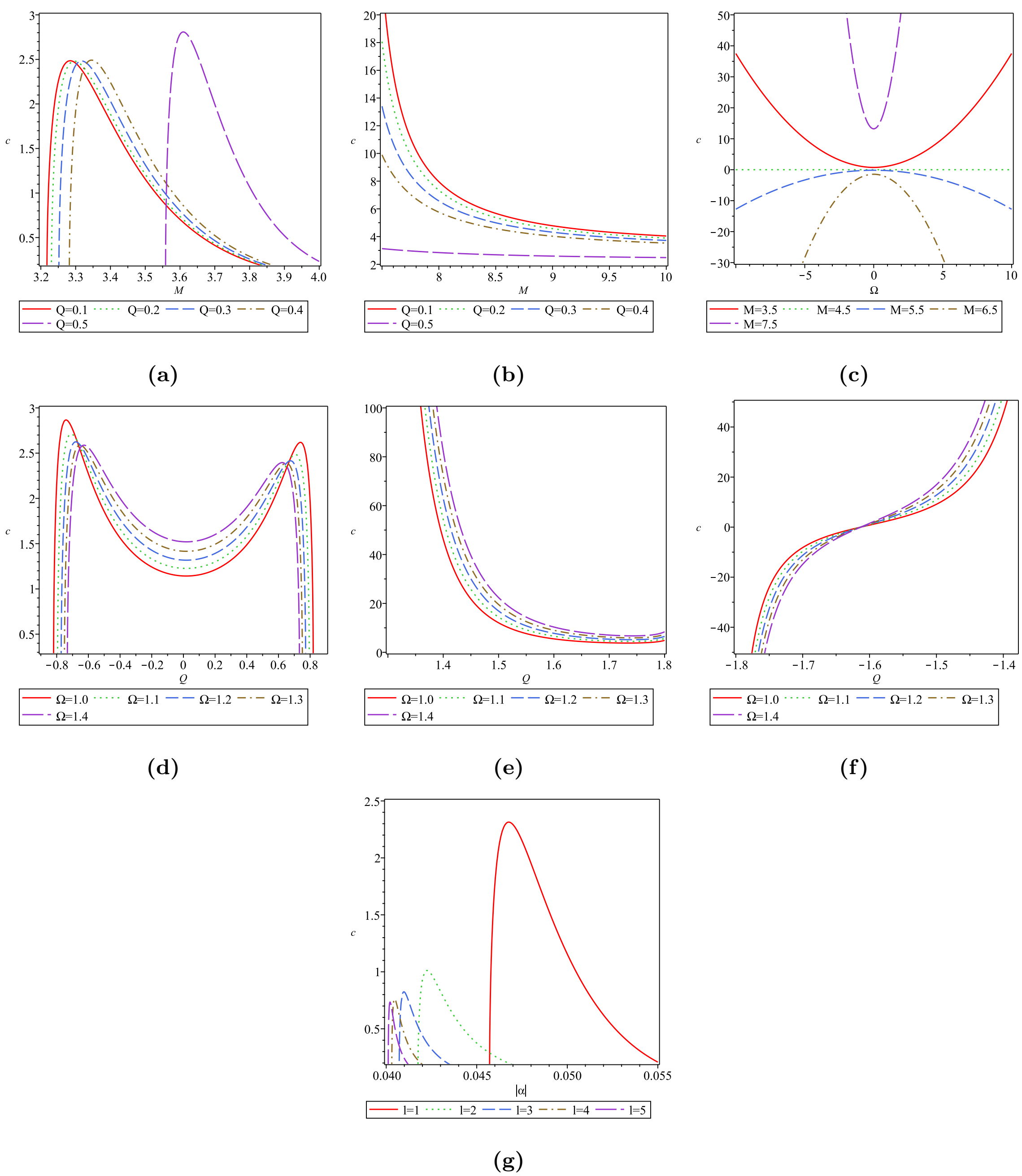


Figure 5.11: c as a function of (a) and (b) M, with $\Omega = 1,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of Q. (c) Ω, with $Q = 0.1,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of M. (d), (e), and (f) Q, with $M = 3.5,\ l = 1,\ |\alpha| = 0.05$, and several numerical values of Ω. (g) $|\alpha|$, with $\Omega = 1,\ Q = 0.1, M = 3.5$, and several numerical values of l. In these plots, we set, $L = 1$, and $r_+ = 1$.

6 Conclusion

In this thesis, we reviewed the Kerr/CFT correspondence for both extremal and non-extremal Kerr black holes. We extended the Kerr/CFT paradigm to the case of rotating black holes in Brans-Dicke theory and $f(T)$ gravity theory. In chapter 2, we started by deriving the vacuum Einstein field equation of general relativity. We arrived at the Kerr solution, which is the exact rotating body solution to the vacuum Einstein equation, and its thermodynamics properties. We found that the macroscopic entropy of a Kerr black hole is simply proportional to the area of the event horizon. Following this, we reviewed the teleparallel gravity which is the foundation of $f(T)$ gravity. We started by reviewing the torsion tensor and deriving the field equation of teleparallel gravity which is equivalent to the field equation of general relativity. Then, we generalized the theory by replacing the torsion tensor in the theory with an arbitrary function $f(T)$. Varying the action of $f(T)$ gravity led us to derive the charged AdS solution in cylindrical coordinates and also its rotating solution. We found that the first law of black hole thermodynamics in $f(T)$ gravity is violated. In the limit of $f(T) \ll 1$, however, the first law of black hole thermodynamics in $f(T)$ is recovered and the entropy reads $S = f'(T)A/4$, where A is the area of the event horizon.

The study of AdS/CFT and Kerr/CFT correspondence has attracted lots of attention on the research of conformal field theory in the past two decades. In chapter 3, we reviewed the conformal field theory starting from the higher dimensional CFT and then 2D CFT. Two-dimensional CFT is a system that has $SL(2,R)_L \times SL(2,R)_R$ symmetries. The generators of 2D CFT obey the commutation relation which is called the Witt algebra. This algebra can be extended by introducing the quantity named the central charge. The extended version is then called the Virasoro algebra. With the central charge and the energy of the 2D CFT, we can compute the thermal Cardy entropy of the 2D CFT, which is the formula that can be used to produce the microscopic entropy of the Kerr black hole.

The isometries of AdS$_3$ space form the group $SL(2,R)_L \times SL(2,R)_R$. The near-horizon geometry of an extremal Kerr black hole has an AdS-like structure. Since AdS$_3$ and 2D CFT have the same $SL(2,R)_L \times SL(2,R)_R$ symmetry, it was proposed that the extremal Kerr black hole also has a holographic relation to the 2D CFT. In chapter 4, we lightly reviewed the Kerr/CFT correspondence for the extremal Kerr black hole. The macroscopic Bekenstein-Hawking entropy of extremal Kerr was reproduced by calculating the Cardy entropy formula in 2D CFT. It was surprising that a formula in conformal field theory can reproduce the result of a gravitational theory. In this thesis, we restricted our attention to one type of observable in Kerr/CFT which are the thermodynamics quantities. Another example of an observable that we did not include in this work is the absorption cross section for scalar, photons, and gravitons scattering calculated from the gravitational theory and the 2D CFT correlation function. The conformal symmetry, however, can not be identified on the near-horizon region of non-extremal Kerr black holes since it is not apparent from the geometry, but depends on how we probe the black hole. It was found that the $SL(2,R)_L \times SL(2,R)_R$ symmetry can be found in the near region scalar field probe around the black hole. To be precise, the radial equation of the scalar wave function could be written as the $SL(2,R)_L \times SL(2,R)_R$ Casimir eigen-equation. The conformal symmetry is spontaneously broken under the angular identification $\phi + 2\pi$, which suggests that the non-extremal Kerr black hole is dual to the finite temperature of the 2D CFT. Subsequently, the Kerr/CFT correspondence was established by matching the Cardy entropy and the Bekenstein-Hawking entropy.

In chapter 5, we first attempted to extend the concept of black hole holography to the case of black holes in Brans-Dicke theory. The rotating Janis-Newman-Winicour black hole is the Kerr-like solution in the Brans-Dicke theory but in the Einstein frame. Unfortunately, our attempt was unsuccessful since we found that the rotating Janis-Newman-Winicour solution is invalid. Another attempt to find a non-trivial Kerr-like solution in the Brans-Dicke theory has been made. The solution is called the Brans-Dicke-Kerr black hole. Yet again, this attempt to establish the Kerr/CFT correspondence in this theory was also unsuccessful. The scalar wave equation in the background of the Brans-Dicke-Kerr black hole can not be separated into the radial and the angular parts. As we were aware, for a non-extremal black hole, the $SL(2,R)_L \times SL(2,R)_R$ symmetry can be found in the radial equation of the scalar probe around the black hole. Thus, we conclude that the inseparability of the scalar wave equation eliminates the possibility of any holography aspect for Brans-Dicke-Kerr black hole.

Our third attempt was extending the concept of black hole holography to the non-extremal 4D rotating charged AdS black holes in $f(T)$-Maxwell theory. We explicitly construct the hidden conformal symmetry for the rotating black holes in $f(T)$-Maxwell theory with a negative cosmological constant. We mainly consider the near-horizon region, as the metric function which determines the event horizon is a triple-quadratic equation. In this region, we show that the radial equation of the scalar wave function could be written as the $SL(2,R)_L \times SL(2,R)_R$ squared Casimir equation, indicating a *local* hidden conformal symmetry acting on the solution space. The conformal symmetry is spontaneously broken under the angular identification $\phi \sim \phi + 2\pi$, which suggests the rotating charged AdS black holes in quadratic $f(T)$ gravity should be dual to the finite temperatures T_L and T_R in the 2D CFT. Instead of calculating the central charges using the asymptotic symmetry group, we calculated the central charges by assuming the Cardy entropy for the dual CFT matches the macroscopic Bekenstein-Hawking entropy. These results suggest that rotating charged AdS black holes in quadratic $f(T)$ gravity with particular values of M, Ω, Q, and $|\alpha|$, are dual to a 2D CFT.

It is an open question to find the near-horizon (near-)extremal geometry of the rotating charged AdS spacetime in quadratic $f(T)$ gravity. In the future, we may calculate the central charges using the asymptotic symmetry group to confirm our results in this thesis. We can also study various kinds of superradiant scattering off the near-extremal black hole as potential evidence to support the holographic picture for the non-extremal 4D rotating charged AdS black holes in $f(T)$-Maxwell gravity.

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